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Truncated Realized Covariance when prices have infinite variation jumps

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Abstract

The speed of convergence of the truncated realized covariance to the integrated covariation between the two Brownian parts of two semimartingales is heavily influenced by the presence of infinite activity jumps with infinite variation. Namely, the two processes small jumps play a crucial role through their degree of dependence, other than through their jump activity indices. This theoretical result is established when the semimartingales are observed discretely on a finite time horizon. The estimator in many cases is less efficient than when the model only has finite variation jumps.

The small jumps of each semimartingale are assumed to be the small jumps of a Lévy stable process, and to the two stable processes a parametric simple dependence structure is imposed, which allows to range from independence to monotonic dependence.

The result of this paper is relevant in financial economics, since by the truncated realized covariance it is possible to separately estimate the common jumps among assets, which has important implications in risk management and contagion modeling.

Keywords: Brownian correlation coefficient, integrated covariation, co-jumps, Lévy copulas, threshold estimator.
Jel classification: C13, C14, C58

1 Introduction

We consider two state variables evolving as follows

$$\begin{aligned} dX_t^{(1)} &= a_t^{(1)}dt + \sigma_t^{(1)}dW_t^{(1)} + dZ_t^{(1)}, \\ dX_t^{(2)} &= a_t^{(2)}dt + \sigma_t^{(2)}dW_t^{(2)} + dZ_t^{(2)}, \quad t \in [0, T] \end{aligned} \tag{1}$$

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with T fixed, where $W_t^{(2)} = \rho_t W_t^{(1)} + \sqrt{1-\rho_t^2} W_t^{(3)}$; $W^{(1)} = (W_t^{(1)})_{t\in[0,T]}$ and $W^{(3)} = (W_t^{(3)})_{t\in[0,T]}$ are independent Wiener processes; $Z^{(1)}$ and $Z^{(2)}$ are correlated pure jump semimartingale (SM) processes. Given discrete equally spaced observations $X_{t_i}^{(1)}, X_{t_i}^{(2)}$, $i = 1..n$, in the interval $[0, T]$, with $t_i = ih, h = \frac{T}{n}$, we are interested in the identification of the *Integrated Covariation* $IC \doteq \int_0^T \rho_t \sigma_t^{(1)} \sigma_t^{(2)} dt$. It is well known that, as the observation step h tends to 0, the *Realized Covariance* $\sum_{i=1}^n \Delta_i X^{(1)} \Delta_i X^{(2)}$, where $\Delta_i X^{(m)} \doteq X_{t_i}^{(m)} - X_{t_{i-1}}^{(m)}$, converges to the global quadratic covariation $[X^{(1)}, X^{(2)}]_T = \int_0^T \rho_t \sigma_t^{(1)} \sigma_t^{(2)} dt + \sum_{0\le t\le T} \Delta Z_t^{(1)} \Delta Z_t^{(2)}$, where $\Delta Z_t^{(m)} = Z_t^{(m)} - Z_{t-}^{(m)}$, containing also the *co-jumps* $\Delta Z_t^{(1)} \Delta Z_t^{(2)}$. It is also well known that the *Threshold Realized Covariance*, or *Truncated Realized Covariance*,

$$\hat{IC} = \sum_{i=1}^n \Delta_i X^{(1)} I_{\{(\Delta_i X^{(1)})^2 \le r_h\}} \Delta_i X^{(2)} I_{\{(\Delta_i X^{(2)})^2 \le r_h\}},$$

with e.g. $r_h = h^{2u}$, and $u \in (0, 1/2)$, is consistent to IC ([16], [8])[1]. Further, a CLT for $\hat{IC}$ has been established when the jumps processes have *finite jump activity* (FA), i.e. only a finite number of jumps can occur, along each path, in each finite time interval (see [16]), or when the jumps processes have infinite activity (IA) but *finite variation* (FV), i.e. $\sum_{s\le T} |\Delta X_s^{(m)}| < \infty$ a.s., for both $m = 1, 2$ (see [9], Thm 7.4), meaning that the jump activity of the processes is moderate. Namely, the estimator is asymptotically mixed Gaussian and converges with speed $\sqrt{h}$.

In [16] the estimator has been compared in efficiency with other two known estimators of IC; it has been used to estimate the sum of the cojumps of $X^{(1)}$ and $X^{(2)}$ as well as each single cojump; and it has been studied in the presence of irregular sampling and non synchronous data; in [4] and in the web appendix of [16] the finite sample performance of $\hat{IC}$ has been evaluated on simulated data. Similarly as in [15], $\hat{IC}$ tends to zero in the presence of microstructure noises in the data.

Here we are interested in investigating the speed of convergence of $\hat{IC}$ in the case where at least one jump component has infinite variation (iV). This was not known up to now. We find that the speed crucially depends on the small jumps, namely it is determined not only by the jump activity indices of the two components $X^{(m)}$, but also on the dependence degree of their small jumps. In the univariate case the speed found here reduces to the one in [14].

The optimal speed in estimating IC is not known when the jumps have infinite variation. In the univariate case IC becomes the *integrated variance* IV of X, and in [10] Jacod and Reiss have shown that, defined the class $\mathcal{S}_A^r$ of the Ito semimartingales X such that a.s. $\sup_{s\le T} |a_s| + \sup_{s\le T} |\sigma_s^2| + \sup_{s\le T} \int (|\gamma(\omega, x, s)|^r \wedge 1)\nu(dx) \le A$, with $r \in (1, 2]$ and $A \in \mathbb{R}$, the quantity $\rho_h = (h/|\log h|)^{(1-r/2)}$ is the highest possible speed, for any estimator of IV, to be a uniform bound for the models within $\mathcal{S}_A^r$, and the bound is sharp. For a comparison with the truncated estimator, note that when the model has α stable small jumps (as in [14]) then (if A is sufficiently large) it belongs to $\mathcal{S}_A^r$ for any $r > \alpha$, but not to $\mathcal{S}_A^\alpha$, and for any such r we have $\rho_h > (h/|\log h|)^{(1-\alpha/2)}$. Now by taking threshold function $r(h) =$

[1]For the literature on non parametric inference for the IC of stochastic processes driven by Brownian motions plus jumps, see [16].

$(h/|\log h|)^{2u}$, with $u \in (0, 1/2)$, rather than h^{2u}, then the threshold estimator reaches speed $(h/|\log h|)^{2u(1-\alpha/2)}$, which is the same as ρ_h as soon as $2u = (1 - r/2)/(1 - \alpha/2) < 1$.

In [11] Jacod and Todorov refine an estimator given in [10] and show that its speed is $\sqrt{h}$ in the semiparametric class, that we call $\mathcal{S}^{loc}_{Stab}$, of the Ito semimartingales X having α stable-like small jumps, Ito semimartingale volatility σ and coefficients with a specified paths regularity.

Now, given an estimator $\tilde{IV}$, a possible estimator of IC is given by $\tilde{IV}(X^{(1)} + X^{(1)})/2 - \tilde{IV}(X^{(1)})/2 - \tilde{IV}(X^{(1)})/2$, thus the best convergence speed of an estimator of IC is bounded by ρ_h if the model falls within $\mathcal{S}^r_A$ and is $\sqrt{h}$ if the model falls within $\mathcal{S}^{loc}_{Stab}$. The univariate version of the semiparametric model we are considering in this paper is not necessarily included in $\mathcal{S}^{loc}_{Stab}$, because we have a general càdlàg process σ. However we remark that the speed of $\hat{IC}$ we show below: in cases (10) is $\sqrt{h}$, so it is optimal and better than ρ_h, moreover the asymptotic variance of $(\hat{IC} - IC)/\sqrt{h}$ is the optimal $2\int_0^T \sigma_s^4 ds$, as in the case of symmetric jumps in [11], but is better than in the case of not symmetric jumps in [11]; in cases (12) the speed of $\hat{IC}$ is worse than $\sqrt{h}$ but is better than ρ_h; while in cases (11) the speed is worse than both.

Estimation of IC is of strong interest both in financial econometrics (see e.g. [3]) and for portfolio risk and hedge funds management ([6]), in particular $[X^{(1)}, X^{(2)}] - \hat{IC}$ gives a tool for measuring the propagation among assets of effects due to important negative or positive economic events.

An outline of the paper is as follows. In section 2 we illustrate the framework, in section 3 we establish the exact convergence speed when both the $Z^{(m)}$ have IA and at least one has iV. Namely, we assume that the small jumps of the $Z^{(m)}$ are stable and their dependence degree can range, in a specified way, from independence to monotonic dependence. The proofs of Theorems 3.1 and 3.2 are contained in Appendix 1, while Appendix 2 contains the proofs of the needed auxiliary results which are stated in section 3 and Appendix 1.

2 The framework

Given a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$, let $X^{(1)} = (X_t^{(1)})_{t\in[0,T]}$ and $X^{(2)} = (X_t^{(2)})_{t\in[0,T]}$ be two real processes defined by (1) and $X_0 = (0, 0)$, where

A1. *the coefficients* $\sigma^{(m)} = (\sigma_t^{(m)})_{t\in[0,T]}$, $a^{(m)} = (a_t^{(m)})_{t\in[0,T]}$, $m = 1, 2$, *and* $\rho = (\rho_t)_{t\in[0,T]}$ *are adapted càdlàg processes,*

A2. *for* $m = 1, 2$, $Z^{(m)} = J^{(m)} + M^{(m)}$ *are jump Ito SMs, with*

$$J^{(m)} \doteq \int_0^\cdot \int_{\{|\gamma^{(m)}(\omega,x,s)|>1\}} \gamma^{(m)}(\omega, x, s)\mu^{(m)}(\omega, dx, ds),\ M^{(m)} \doteq \int_0^\cdot \int_{\{|\gamma^{(m)}(\omega,x,s)|\le 1\}} \gamma^{(m)}(\omega, x, s)\tilde{\mu}^{(m)}(\omega, dx, ds),$$

where, for each $m = 1, 2$, $\mu^{(m)}$ *is the Poisson random measure counting the jumps of* $Z^{(m)}$ *and* $\tilde{\mu}^{(m)}(\omega, dx, ds) \doteq \mu^{(m)}(\omega, dx, ds) - \nu^{(m)}(dx)ds$ *is its compensated measure (see [9]).*

It turns out that $J^{(m)}$ are FA jump processes; they account for the rare and large (with size bigger in absolute value than 1) jumps of $X^{(m)}$. On the contrary, $M^{(m)}$ have generally

IA jumps (the path ω of $M^{(m)}$ jumps infinitely many times on $[0,t]$ iff $\int_0^t \int_{\{|\gamma^{(m)}(\omega,x,s)|\leq 1\}} \nu^{(m)}(dx)$ $ds = \infty$); $M^{(m)}$ are compensated sums of very frequent and small jumps.

For each $n \in \mathbb{N}$ we observe $X^{(1)}, X^{(2)}$ discretely and synchronously at times $t_i = ih$. Since $h = T/n$, then $h \to 0$ iff $n \to \infty$.

A3. *We choose a deterministic function* r_h *of* h*, called* threshold*, satisfying*

$$\lim_{h\to 0} r_h = 0, \quad \lim_{h\to 0} \frac{hlog\frac{1}{h}}{r_h} = 0.$$

Denote, for each $m = 1, 2$, by

$$D_t^{(m)} = \int_0^t a_s^{(m)} ds + \int_0^t \sigma_s^{(m)} dW_s^{(m)}, \quad Y_t^{(m)} = D_t^{(m)} + J_t^{(m)}$$

respectively the Brownian semimartingale part (BSM) of $X^{(m)}$ and the BSM part plus the FA jump component.

The truncated realized covariance is able to separately capture IC because it excludes from $\sum_{i=1}^n \Delta_i X^{(1)} \Delta_i X^{(2)}$ those increments where jumps bigger than the threshold occurred, so when $h \to 0$ all the jumps are excluded (see point iii) in the proof of Theorem 1 in [13], then Lemma A.2 in [5] and Lemma 1 in [13]). However, the remaining jumps, that in absolute value are below $\sqrt{r_h}$, determine, in some cases, the speed of convergence of $\hat{IC}$.

Notations. Given two (possibly random) sequences U_n, V_n, we say that $U_n = O_P(V_n)$ if for any $\epsilon > 0$ there exists a constant $\eta > 0$ and an $\bar{n}$ such that for all $n \geq \bar{n}$, $P(|U_n| > \eta |V_n|) < \epsilon$. We write $U_n \sim V_n$ when as $n \to \infty$ we have both $U_n = O_P(V_n)$ and $V_n = O_P(U_n)$. When $\forall n,\ a.s. V_n \neq 0 : U_n = O_P(V_n)$ means that, for sufficiently large n, the sequence U_n/V_n is bounded in probability (i.e. tight); with a a constant, $U_n \approx aV_n$ means that $U_n/V_n \stackrel{P}{\to} a$, with $\stackrel{P}{\to}$ denoting convergence in probability; $U_n << V_n$ means that $U_n/V_n \stackrel{P}{\to} 0$; $U_n >> V_n$ means that $U_n/V_n \stackrel{P}{\to} +\infty$.

3 Main results

We find here the speed of convergence of $\hat{IC} - IC$ to 0 when both $M^{(m)} \neq 0$ and at least one of them has iV. We specialize our analysis to the case where the small jumps of each $X^{(m)}$ are stable, i.e. $M_t^{(m)} = L_t^{(m)} - z^{(m)}t - \sum_{s\leq t} \Delta L_s^{(m)} I_{\{|\Delta L_s^{(m)}|>1\}}$, where $L^{(m)}$ are α_m-stable Lévy processes with characteristic triplets $(z^{(m)}, 0, \nu^{(m)}(dx))$, with $\nu^{(m)}$ given below. Further, we assume that the occurrence of the joint jumps of $L^{(1)}$ and $L^{(2)}$ is characterized by a Lévy copula C ranging in a given class. We have $\alpha_m \in]0,2[$ for each $m = 1, 2$ and assume without loss of generality (wlg) $\alpha_1 \leq \alpha_2$. Since we are interested in the case where at least one $\alpha_m \geq 1$, we assume $\alpha_2 \geq 1$. Further, for simplicity, but wlg, we develop our proofs for the case where the Lévy measure of each $L^{(m)}$ is one sided, i.e. $L^{(m)}$ only makes jumps with positive sizes.

A4. *Take* $\alpha_2 \geq 1$, *and* $\alpha_1 \in (0, \alpha_2]$. *With* $c_m > 0$, $m = 1, 2$, *the jumps of each* $L^{(m)}$ *have Lévy measure*

$$\nu^{(m)}(dx_m) = c_m x_m^{-1-\alpha_m} I_{\{x_m>0\}} dx_m.$$

We denote, for each $m = 1, 2$, by

$$U_m(x_m) := \nu^{(m)}\big([x_m, +\infty[\big) = c_m \frac{x_m^{-\alpha_m}}{\alpha_m}, \quad x_m > 0 \tag{2}$$

the tail integral of the marginal Lévy measure $\nu^{(m)}$ of the jumps of $L^{(m)}$. Note that α_m is the *Blumenthal-Getoor index* of $L^{(m)}$, of $M^{(m)}$ and of $X^{(m)}$.

In order to describe the joint jumps, we make use of Lévy copulas, because, due to the stationarity of the Lévy processes increments, the Lévy copulas allow to separate the time component in the law of a bivariate pure jump Lévy process L from the jump sizes component and allow to describe the dependence between $L^{(1)}$ and $L^{(2)}$ through only the dependence of their jump sizes. Lévy copulas were introduced in [18], further studied in [12] and their properties are well summarized in [6].

A5. *For any* t *the joint jumps occurrence of* $\big(L_t^{(1)}, L_t^{(2)}\big)$ *is described by the following tail integrals*

$$U(x_1, x_2) = \nu_\gamma([x_1, +\infty) \times [x_2, +\infty)) = C_\gamma(U_1(x_1), U_2(x_2))$$

where $C_\gamma(u, v)$ *is a Lévy copula of the form*

$$C_\gamma(u, v) = \gamma C_\perp(u, v) + (1 - \gamma) C_\parallel(u, v),$$

where $C_\perp(u, v) = uI_{\{v=\infty\}} + vI_{\{u=\infty\}}$ *is the independence copula,* $C_\parallel(u, v) = u \wedge v$ *is the total positive dependence copula, and* γ *ranges in* $[0, 1]$.

A5 means that, at any t, $(L^{(1)}, L^{(2)})$ can only have two basically different classes of jumps: i) the disjoint ones, meaning that L_t jumps with size either $(0, x_2)$ or $(x_1, 0)$. This type of jumps is regulated only by $C_\perp$; ii) the joint ones, meaning that L_t jumps with size falling into a point (x_1, x_2) with both $x_m \neq 0$. This type of jumps is regulated only by $C_\parallel$, which characterizes a bivariate jump Lévy process $\bar{L}$ whose marginals $L^{(m)}$ are Lévy and only make joint jumps which are completely positively monotonic, i.e. there exists a strictly increasing, strictly positive function f: $\forall s > 0$, $\Delta\bar{L}_s^{(2)} = f(\Delta\bar{L}_s^{(1)})$. In fact the sizes (x_1, x_2) realized by the jumps of $\bar{L}_s$ turn out to be supported by the graph of $f(x_1) = U_2^{-1}(U_1(x_1))$, which in our case of one sided α-stable marginals is given by $f(x_1) = ((c_1\alpha_2)/(\alpha_1 c_2))^{-1/\alpha_2} x_1^{\alpha_1/\alpha_2}$.

Our assumption that L has Lévy measure ν_γ means that its jumps on the set given by the union of the graph of f and the positive sides of the Cartesian axes. Each marginal $\mu^{(m)}$ counts the projection on axis x_m of *all* the realized jumps of L. However when a realized jump x_1 is so that there exists a realized x_2 such that $x_2 = f(x_1)$ then x_1 is interpreted as the first component of a joint jump. Any other types of jump of $L^{(1)}$ are interpreted as being associated to a zero complementary component, i.e. as being the projection of a disjoint jump (and analogously for $L^{(2)}$). By changing γ we keep the same marginals $L^{(m)}$ and the same joint or disjoint jumps, but we change the weight given to the different classes of jumps by the underlying probability measure. Process $\bar{L}$ has joint Lévy measure $\nu_{\parallel}([x_1,+\infty)\times[x_2,+\infty)) = I_{\{x_1\neq 0, x_2\neq 0\}}\nu^{(1)}([x_1\vee f^{-1}(x_2),+\infty))$, so the ν_γ defined by **A5** is equivalently writable as $\nu_\gamma([x_1,+\infty)\times[x_2,+\infty)) =$

$$\gamma I_{\{x_2=0\}}\nu^{(1)}([x_1,+\infty))+\gamma I_{\{x_1=0\}}\nu^{(2)}([x_2,+\infty))+(1-\gamma)I_{\{x_1\neq 0,x_2\neq 0\}}\nu^{(1)}([x_1\vee f^{-1}(x_2),+\infty)). \tag{3}$$

Remarks. i) **A5** is equivalently expressed by:

$$L^{(m)} = L'^{(m)} + \bar{L}^{(m)}, \quad m = 1, 2,$$

where $L'^{(m)}$ has triplet $(z'^{(m)}, 0, \gamma\nu^{(m)}(dx))$, $m = 1, 2$, $\bar{L}^{(1)}$ has $(\bar{z}^{(1)}, 0, (1-\gamma)\nu^{(1)}(dx))$, $(L'^{(1)}, L'^{(2)}, \bar{L}^{(1)})$ are independent while, as said, $\Delta\bar{L}_s^{(2)} = f(\Delta\bar{L}_s^{(1)})$. In particular **A5** is satisfied when the bivariate jumps $\mathbf{Z}$ follow a *factor model*

$$Z^{(1)} = V^{(1)}, Z^{(2)} = aV^{(2)} + bV^{(1)},$$

with $V^{(1)}, V^{(2)}$ independent pure jump Lévy processes, and $a, b \in \mathbb{R}$: $\bar{L} = (V^{(1)}, bV^{(1)})$ and $f(x) = bx$.

ii) Note that in our framework the two components of $\bar{L}$ have the *same* number of jumps, however they can have different jump indices α_m. In a model with $\Delta_t\bar{L}^{(2)} = f(\Delta_t\bar{L}^{(1)})$ but $f(x)\neq bx$, $\bar{L}^{(1)}$ could make jumps much smaller than $\bar{L}^{(2)}$, implying $\bar{\alpha}_1 < \bar{\alpha}_2$. When instead $f(x) = bx$ then the two $\bar{L}^{(m)}$ have the same jump activity index.

The processes we chose to deal with are quite representative since in fact many commonly used models in finance (Variance Gamma model, CGMY model, NIG model, etc.) have Lévy measures related to the ones in assumption **A4**, in the sense that they are tempered stable processes where the order of magnitude of the tail integrals as $x_m \to 0$ is as in (2). Moreover C_γ allows to range from a framework of independent jumps components to a framework where the components are completely positively monotonic.

The speed of convergence of $\hat{IC} - IC$ is strictly related to the speed of convergence to zero of the sum of the small co-increments $\Delta_i M^{(1)} I_{|\Delta_i M^{(1)}|\leq\sqrt{r_h}}\Delta_i M^{(2)} I_{|\Delta_i M^{(2)}|\leq\sqrt{r_h}}$ (as it happened in [14] for the univariate case), which substantially behaves like the sum of the small co-jumps $\sum_{s\leq T}\Delta M_s^{(1)} I_{|\Delta M_s^{(1)}|\leq\sqrt{r_h}}\Delta M_s^{(2)}\ I_{|\Delta M_s^{(2)}|\leq\sqrt{r_h}}$ (see [2], Lemma 5), whose expectation is $T\int_{0\leq x,y\leq\sqrt{r_h}} xy\nu_\gamma(dx,dy)$. Note that, as soon as $\varepsilon < 1$, in restriction to the set of jump sizes $(0,\varepsilon]\times(0,\varepsilon]$, the jumps of the bivariate processes $\mathbf{M}$ and $\mathbf{L}$ coincide. We need assumption **A5** in order to control the speed of convergence to zero of integrals like $\int_{0\leq x,y\leq\varepsilon} x^k y^m\ \nu_\gamma(dx,dy)$, for $\varepsilon > 0$ and integers k, m.

In our main Theorem (Theorem 3.2) we are going to show that

$$\hat{IC} - IC \sim \sqrt{h}U_h + \sum_{i=1}^n \xi_i + \sum_{i=1}^n \int_{t_{i-1}}^{t_i}\sigma_s^{(1)}dW_s^{(1)}\int_{t_{i-1}}^{t_i}\sigma_s^{(2)}dW_s^{(2)} I_{\{\sum_{s\in]t_{i-1},t_i]} I_{\{|\Delta M_s^{(2)}|>\sqrt{r_h}\}}\geq 1\}}, \tag{4}$$

where

$$\xi_i = \xi_i^\varepsilon \doteq \Delta_i M'^{(1)}\Delta_i M'^{(2)},$$

and for $m = 1, 2$

$$M_t'^{(m)} \doteq M_t^{(m)} - \sum_{s\leq t}\Delta M_s^{(m)} I_{\{|\Delta M_s^{(m)}|>\varepsilon\}} = \int_0^t\int_{\{0<x\leq\varepsilon\}} x\tilde{\mu}^{(m)}(dx,ds) - t\int_{\{\varepsilon<x\leq 1\}} x\nu^{(m)}(dx),$$

and U_h is a sequence of rvs converging stably in law to a mixed Gaussian rv. So we preliminarily state the following crucial result, which deals with the asymptotic behavior of $\sum_{i=1}^{n} \xi_i$.

Theorem 3.1. *Assume* $0 < \alpha_1 \le \alpha_2 < 2$, $\alpha_2 \ge 1$, $0 < c_1 \le c_2$, $\varepsilon = \sqrt{r_h} = h^u$, $u \in (0, \frac{1}{2})$. *As* $h \to 0$ *we have*
i) if $\gamma \in [0,1)$, *then for any choice of* α_1, α_2 *and* u *as in the assumptions:*

$$\sum_i \xi_i \approx nE[\xi_1] \approx T(1-\gamma)C(1,1)\varepsilon^{1+\frac{\alpha_2}{\alpha_1}-\alpha_2} I_{\{\alpha_1>\alpha_2 u\}\cup\{\alpha_1=\alpha_2 u,\alpha_2>1\}} + Thc_{A_1}c_{A_2}F_0(\varepsilon),$$

where $F_0(\varepsilon) = -\varepsilon^{1-\alpha_2} I_{\{\alpha_1 \le \alpha_2 u, \alpha_2>1\}} + \log\frac{1}{\varepsilon} I_{\{\alpha_1\le\alpha_2 u,\alpha_2=1\}}$; $C(k,m) \doteq c_2\left(\frac{\alpha_2 c_1}{\alpha_1 c_2}\right)^{\frac{k}{\alpha_1}} \frac{1}{m+\frac{\alpha_2}{\alpha_1}k-\alpha_2} > 0$, *for* $k, m \ge 0$; *and, for* $m = 1, 2$, $c_{A_m} \doteq \frac{c_m}{1-\alpha_m} I_{\alpha_m\neq 1} + c_m I_{\alpha_m=1}$;
ii) if $\gamma = 1$ *but* $(\alpha_1,\alpha_2) \neq (1,1)$, *and: if* $\{\alpha_1 < 1, \alpha_2 \ge 1\} \cup \{\alpha_1 = 1 < \alpha_2\}$ *we take* $u \in (\frac{1}{2+\alpha_2-\alpha_1}, \frac{1}{2})$; *while if* $\{1 < \alpha_1 \le \alpha_2\}$ *we take* $u \in (\frac{1}{\alpha_1+\alpha_2}, \frac{1}{2})$; *then we have*

$$\sum_i \xi_i \approx nE[\xi_1] \approx Thc_{A_1}c_{A_2}F_1(\varepsilon),$$

where $F_1(\varepsilon) = -\varepsilon^{1-\alpha_2} I_{\{\alpha_1<1<\alpha_2\}} + \log\frac{1}{\varepsilon} I_{\{\alpha_1<1=\alpha_2\}} - \varepsilon^{1-\alpha_2}\log\frac{1}{\varepsilon} I_{\{\alpha_1=1<\alpha_2\}} + \log^2\frac{1}{\varepsilon} I_{\{\alpha_1=\alpha_2=1\}} +$ $+\varepsilon^{2-\alpha_1-\alpha_2} I_{\{1<\alpha_1\le\alpha_2\}}$;
iii) if $\gamma = \alpha_1 = \alpha_2 = 1$, *for any* $u \in (0, \frac{1}{2})$: *with* $C_m(k) = \frac{c_m}{k-\alpha_m}$, *for* $k, m = 1, 2$, *we have*

$$\sum_i \xi_i \approx \sqrt{n\ Var(\xi_1)}\ U_h \approx \sqrt{h}\varepsilon\sqrt{TC_1(2)C_2(2)}\ U_h.$$

Remarks. i) Since $c_{A_m} > 0$ for $\alpha_m \le 1$ while $c_{A_m} > 0$ for $\alpha_m > 1$ and within F_0 we always have $\alpha_1 < 1$, then we always have $c_{A_1}c_{A_2}F_i(\varepsilon) > 0$, $i = 0, 1$.

ii) As for ii) above, if either $\alpha_1 < 1$ or $\alpha_1 = 1 < \alpha_2$ then we have $\alpha_1 < \alpha_2$ and requiring that $u > 1/(2+\alpha_2-\alpha_1)$ is possible because $1/(2+\alpha_2-\alpha_1) < 1/2$. On the contrary, the set $\{1 < \alpha_1 \le \alpha_2\}$ contains the case $\alpha_1 = \alpha_2$ in which $u > 1/(2+\alpha_2-\alpha_1) = 1/2$ is not admissible. Note that condition $u > 1/(2+\alpha_2-\alpha_1)$ implies $u > 1/(\alpha_1+\alpha_2)$ when $\alpha_2 > \alpha_1 > 1$.

iii) The speed of convergence of $\sum_i \xi_i$ is determined not only by each α_1, α_2 but also by the degree γ of dependence of the two small jumps components of $\mathbf{Z}$.

iv) We have that $\sum_{i=1}^{n} \xi_i$ tends to zero much faster when $\gamma = 1$ than when $\gamma \in [0,1)$ (we obtain that by using Proposition 4.4 and comparing $nE[\xi_1]$ in (30) with $nE[\xi_1]$ or $\sqrt{nVar(\xi_1)}$ in (31), while matching all the sets of (α_1, α_2)). In other words, the speed at which the sum of the co-increments ξ_i tends to zero is much faster when $M^{(1)}, M^{(2)}$ are independent, in fact ξ_i is led by the small co-jumps and in the independent case the sum of the small co-jumps is zero (rather than being small).

v) Comparing the speed of $\sum_i \xi_i$ with $\sqrt{h}$, we reach that $\sum_i \xi_i << \sqrt{h}$ substantially when α_1 is sufficiently small (and still $\alpha_2 \ge 1$). In this case the co-increments of $M^{(1)}, M^{(2)}$ are negligible with respect to (wrt) the Browinan co-increments. More precisely, using Proposition 4.4, Theorem 3.1 and (36) in Appendix 2, defined

$$\alpha_1^\star \doteq \frac{\alpha_2 u}{\alpha_2 u - u + 1/2} \in (2u, 1), \quad \alpha_1^{\star\star} \doteq \frac{1+2u(2-\alpha_2)}{2u} > \frac{1}{2u} > 1,$$

we reach (see the proof in Appendix 2) that:

$$\begin{cases} \text{if } \gamma \in [0,1): & \sum_i \xi_i << \sqrt{h} \quad \text{iff } \alpha_1 < \alpha_1^\star; \\ \text{if } \gamma = 1: & \sum_i \xi_i << \sqrt{h} \quad \text{iff } \alpha_1 < \alpha_1^{\star\star}. \end{cases} \tag{5}$$

Since $\alpha_1^\star < 1 < \alpha_1^{\star\star}$, the above result means that when the two small jumps components $M^{(m)}$ are independent, then the impact of their co-increments on the convergence speed of $\hat{IC} - IC$ is negligible, wrt the impact $\sqrt{h}$ of the Browinan co-increments, for a wider range of values α_1. □

Here is the main result of our paper.

Theorem 3.2. *If $\rho \not\equiv 0$ and $\rho, \sigma^{(m)}$, $m = 1, 2$, are s.t. when $h \to 0$,*

$$\forall s \geq t: \; s - t \leq h, \; \textit{then } E[|\sigma_s^{(m)} - \sigma_t^{(m)}|^2] \leq K(s-t), \quad m = 1, 2, \tag{6}$$

with $0 < \alpha_1 \leq \alpha_2 < 2$, $\alpha_2 \geq 1$, $0 < c_1 \leq c_2$, $\varepsilon = \sqrt{r_h} = h^u$, $u > 0$ such that

$$1/2 > u > \begin{cases} \frac{1}{2+\alpha_2-\alpha_1} \vee \frac{1}{3-\frac{\alpha_2}{2}} & \textit{if } \alpha_1 < \alpha_2 \\ \frac{1}{3-\frac{\alpha_2}{2}} & \textit{if } \alpha_1 = \alpha_2 \end{cases} \tag{7}$$

then, as $h \to 0$, we have

$$\hat{IC} - IC \sim \sqrt{h} U_h + \sum_{i=1}^n \xi_i + \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \sigma_s^{(1)} dW_s^{(1)} \int_{t_{i-1}}^{t_i} \sigma_s^{(2)} dW_s^{(2)} I_{\{\sum_{s \in]t_{i-1}, t_i]} I_{\{|\Delta M_s^{(2)}| > \sqrt{r_h}\}} \geq 1\}} \tag{8}$$

$$\sim \sqrt{h} + (1-\gamma)\varepsilon^{1+\frac{\alpha_2}{\alpha_1}-\alpha_2} + h\varepsilon^{-\alpha_2} \tag{9}$$

$$\sim \sqrt{h} \; I_{\{\alpha_2 \in [1, \frac{1}{2u})\}} \left[I_{\{\gamma = 1\}} + I_{\{\gamma \in [0,1), \alpha_1 \leq \alpha_1^\star\}} \right] \tag{10}$$

$$+ \varepsilon^{1+\frac{\alpha_2}{\alpha_1}-\alpha_2} I_{\{\gamma \in [0,1)\}} \left[I_{\{\alpha_1^\star < \alpha_1 \leq \alpha_2 \in [1, \frac{1}{2u})\}} + I_{\{\alpha_2 \geq \frac{1}{2u}\}} I_{\{\alpha_2 = \alpha_1\} \cup \{\alpha_1 < \alpha_2 < \alpha_1(\frac{1}{u}-1)\}} \right] \tag{11}$$

$$+ h\varepsilon^{-\alpha_2} \; I_{\{\alpha_2 \geq \frac{1}{2u}\}} \left[I_{\{\gamma=1\}} + I_{\{\gamma \in [0,1)\}} I_{\{\alpha_1 \frac{1}{u} \leq \alpha_2\} \cup \{\alpha_1(\frac{1}{u}-1) \leq \alpha_2 < \alpha_1 \frac{1}{u}\}} \right]. \tag{12}$$

Remarks on the last result.

i) Condition $\alpha_2 < \alpha_1(1/u - 1)$ is equivalent to $u < \alpha_1/(\alpha_2 + \alpha_1)$ and we did not include it among the ones in (7) because such conditions are required for the convergence of some terms of I_4 (defined within the proof of the Theorem) in $\hat{IC} - IC$, while $\alpha_1/(\alpha_2 + \alpha_1)$ is only a separator to establish whether the leading term is $\varepsilon^{1+\alpha_2/\alpha_1-\alpha_2}$ or $h\varepsilon^{-\alpha_2}$. There is another proof for the convergence of some of the cited terms of I_4, which avoids conditions (7), but it is much longer than the one given in Appendix 1.

ii) Note that $\alpha_1(1/u - 1) \leq \alpha_2$ implies $\alpha_1 < \alpha_2$; $\alpha_1(1/u - 1) > \alpha_2$ implies $\alpha_2 u < \alpha_1$. If $\alpha_1 < \alpha_2$, (7) implies that $u > 1/4$.

iii) Similarly as for $\sum_{i=1}^n \xi_i$, the convergence speed of $\hat{IC} - IC$ depends both on the jump activity indices α_1, α_2 and on the dependence degree γ of the small jumps. This implies that $\hat{IC}$ contains information that we could exploit to estimate such a dependence degree.

Note that when the dependence degree increases (γ decreases) then the leading term of $\sum_{i=1}^n \xi_i$ also increases ($\sum_i E[\xi_i]$ increases and $\sqrt{nVar(\xi_1)} << \sum_i E[\xi_i]$), and the estimation error $\hat{IC} - IC$ increases. An higher leading term of $\sum_i \xi_i$ means that the average weight of the small jumps is higher so that the disturbing noise when estimating the Brownian feature IC is higher. That is: the higher the dependence degree, the higher the disturbing noise.

iv) Basically, when u is close to $1/2$ (i.e. satisfying conditions (7)), if the small jumps are dependent ($\gamma \in [0,1)$), the speed is: $\sqrt{h}$ when α_1, α_2 are small (i.e $\alpha_1 < \alpha_1^\star$ and $\alpha_2 < 1/(2u)$; note that when $\alpha_2 < 1/(2u)$ then $\alpha_1 < \alpha_1^{\star\star}$); $\varepsilon^{1+\alpha_2/\alpha_1-\alpha_2}$ if either the indices have intermediate values (i.e. $\alpha_1^\star < \alpha_1 \le \alpha_2 < 1/(2u)$) or they assume the largest possible values and either they coincide or they are close (i.e. either $\alpha_1 = \alpha_2 \ge 1/(2u)$ or $\alpha_1 < \alpha_2 < \alpha_1(1/u - 1)$ with still $\alpha_2 \ge \frac{1}{2u}$); $h\varepsilon^{-\alpha_2}$ when α_2 is large and the indices are very different (i.e. $\alpha_2 \ge 1/(2u)$ and either $2\alpha_1 < \alpha_1/u \le \alpha_2$ or $\alpha_1 < \alpha_1(1/u-1) \le \alpha_2 < \alpha_1/u$).

If the small jumps are independent ($\gamma = 1$), then the speed is: $\sqrt{h}$ if $\alpha_2 < 1/(2u)$; $h\varepsilon^{-\alpha_2}$ if $\alpha_2 \ge 1/(2u)$.

iv) For $\gamma = 0$ or $\gamma \in (0,1)$ we have the same cases: in the presence of the parallel component, the independent component does not modify the speed of convergence. On the contrary, in the presence of the independent component, the parallel component does worsen the speed of convergence.

v) When the leading term of $\sum_{i=1}^n \xi_i$ is $\sqrt{nVar(\xi_1)} \sim \sqrt{h}\varepsilon^{2-\alpha_1/2-\alpha_2/2}$ $\Big($by Proposition 4.4, e.g. in the case $\gamma = \alpha_1 = \alpha_2 = 1$; or in the case $\gamma = 1$ and $1 < \alpha_1 \le \alpha_2 < 1/(2u)$, since then $u < 1/(2\alpha_2) \le 1/(\alpha_1+\alpha_2)$ $\Big)$ it holds that $\sqrt{nVar(\xi_1)}/\sqrt{h} \to 0$, so $\sum_{i=1}^n \xi_i$ is dominated by $\sqrt{h}$ and the term $\sqrt{h}\varepsilon^{2-\alpha_1/2-\alpha_2/2}$ never appears.

vii) The speed is $\sqrt{h}$ even in some cases with $\alpha_2 \ge 1$ (but $\alpha_2 < 1/(2u)$): any α_1 is, if $\gamma = 1$; for α_1 sufficiently small ($\alpha_1 \le \alpha_1^\star$) if the parallel component is present. In this case we also have a CLT (see below) and in the univariate case the truncated estimator turns out to be efficient.

viii) When $\alpha_1 = \alpha_2 \doteq \alpha \ge 1$ but the two jump components are not necessarily completely monotonic, we reach the following speeds of convergence to zero of $\hat{IC} - IC$: $(1-\gamma)\varepsilon^{2-\alpha}$ if $\gamma \in [0,1)$; $\sqrt{h}$ if $\gamma = 1$ and $\alpha < 1/(2u)$ (note that $\alpha < \alpha_1^\star < 1 \le \alpha_2$ is not in our assumptions); $h\varepsilon^{-\alpha}$ if $\gamma = 1$ and $\alpha \ge 1/(2u)$.

ix) The univariate case is when $\alpha_1 = \alpha_2$ and $\gamma = 0$, and the speed turns out to be $\varepsilon^{2-\alpha} = r_h^{1-\alpha/2}$, for any $\alpha \ge 1$, consistently with [14], where, when $\alpha \ge 1$, the estimation error $\hat{IV} - IV$ for the is led by the IA and iV jump part.

x) For fixed h, the convergence speed is a function $s(\gamma, \alpha_1, \alpha_2, u)$ of our parameters. Such a function is smooth most of the times, however it has some singularities (as is evident in Figure 1; see the details in Appendix 2).

xi) The speed in the worst case scenario is approached when $\gamma \in [0,1)$. Since it is the same for $\gamma \in (0,1)$ or $\gamma = 0$, let us take $\gamma = 0$. For fixed h and u, define R the region identified by the initial assumptions on α_1, α_2 and by (7) and A, B, C the subregions identified respectively in (10), (11) and (12):

$$R = \{(\alpha_1,\alpha_2) : \alpha_1 \in (0,2), \alpha_2 \in [1,2), \alpha_1 \le \alpha_2\} \cap$$

$$\Big(\{\alpha_1 < \alpha_2, \frac{1}{2+\alpha_2-\alpha_1} \vee \frac{1}{3-\frac{\alpha_2}{2}} < u\} \cup \{\alpha_1 = \alpha_2, \frac{1}{3-\frac{\alpha_2}{2}} < u\}\Big),$$

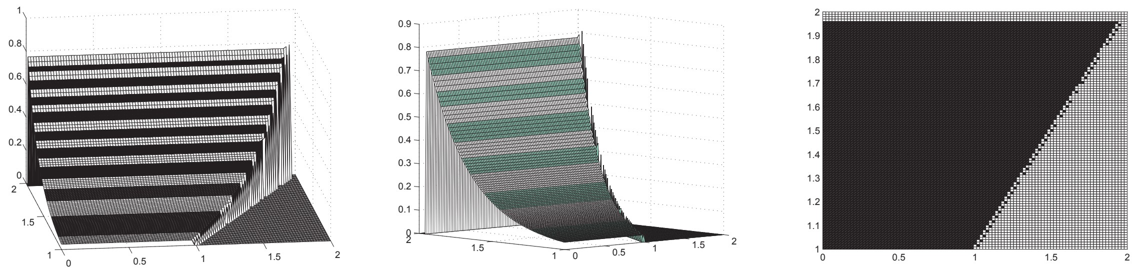

Figure 1: Graph of $s(0,\alpha_1,\alpha_2,u)$ on R, for fixed $h = 1/1000$ and $u = 0.495$, from two different points of view, and region R (black). α_2 varies within the axes with range [1,2], α_1 varies within the one with range [0,2]

$$A = R\cap\{\alpha_1 \le \alpha_1^\star, \alpha_2 < \frac{1}{2u}\}, B = B_1\cup B_2\cup B_3, B_1 = R\cap\{\alpha_1 > \alpha_1^\star, \alpha_2 < \frac{1}{2u}\},$$

$$B_2 = R\cap\{\alpha_1 = \alpha_2, \alpha_2 \ge \frac{1}{2u}\}, B_3 = R\cap\{\alpha_1 < \alpha_2, \alpha_2 \ge \frac{1}{2u},\ \alpha_2 < \alpha_1(\frac{1}{u}-1)\},$$

$$C = R\cap\{\alpha_2 \ge \frac{1}{2u}\}\cap\Big(\{\alpha_1 \le \alpha_2 u\}\cup\{\alpha_2 u < \alpha_1 < \alpha_2,\ \alpha_2 \ge \alpha_1(\frac{1}{u}-1)\}\Big).$$

Then, noting that $1/(3-\alpha_2/2) < u$ iff $\alpha_2 < 2(3-1/u)$, the slowest convergence is approached by

$$\sup_{(\alpha_1,\alpha_2)\in R} s(0,\alpha_1,\alpha_2,u) = \sup_{A\cup B\cup C} s = \sup_{B_1\cup B_2} s = \sup_{\alpha_1=\alpha_2, 1\le\alpha_2<2(3-1/u)} \varepsilon^{1+\frac{\alpha_1}{\alpha_2}-\alpha_2} = \varepsilon^{2(\frac{1}{u}-2)} = h^{2-4u}:$$

note that $h^{2-4u} >> \sqrt{h}$, and the closer is u to 1/2 the slower is the convergence of $\hat{IC}$.

Remark. *When* $\alpha_2 < 1/(2u)$ *and either* $\gamma = 1$ *or both* $\gamma\in[0,1)$ *and* $\{\alpha_1 < \alpha_1^\star\}$, *we have a CLT for* $\hat{IC}-IC$. In fact the only leading term of $\hat{IC}-IC$ is $\sqrt{h}$, which only comes from the components $Y^{(m)}$ of the processes $X^{(m)}$, so the presence of $M^{(1)}$ and $M^{(2)}$ is not influential. Thus using also Theorem 3.4 in [16] and Theorem 4.2 in [7], with $\stackrel{st}{\to}$ denoting stable convergence in law, we have

$$\frac{\hat{IC}-IC}{\sqrt{h}\sqrt{A\hat{V}ar}} \stackrel{st}{\to} \mathcal{N},$$

where $\mathcal{N}$ is a standard Gaussian r.v. and $A\hat{V}ar = h^{1-\frac{r+l}{2}}\sum_{i=1}^n\prod_{m=1}^2(\Delta_i X^{(m)})^2 I_{\{|\Delta_i X^{(m)}|\le\sqrt{r_h}\}}$
$-h^{-1}\sum_{i=1}^{n-1}\prod_{j=0}^1 \Delta_{i+j}X^{(1)} I_{\{|\Delta_{i+j}X^{(1)}|\le\sqrt{r_h}\}}\prod_{j=0}^1\Delta_{i+j}X^{(2)} I_{\{|\Delta_{i+j}X^{(2)}|\le\sqrt{r_h}\}}$,
$\stackrel{P}{\to}\int_0^T(1+\rho_t^2)(\sigma_t^{(1)})^2(\sigma_t^{(2)})^2 dt.$ □

4 Appendix 1

This appendix contains the proofs of Theorems 3.1 and 3.2 and the statements of the necessary tools. We begin with giving the tools to prove Theorem 3.1.

Remark 4.1. *Note that when $k, m \geq 1$ the integral $\int_{0\leq x,y\leq\varepsilon} x^k y^m \nu_\perp(dx, dy)$ is zero, because the independent components of L have no common jumps. It follows that under assumption* **A5***, for both $k \geq 1$ and $m \geq 1$, we have*

$$\int_{0\leq x,y\leq\varepsilon} x^k y^m \nu_\gamma(dx, dy) = (1-\gamma) \int_{0\leq x,y\leq\varepsilon} x^k y^m dC_\parallel(U_1(x), U_2(y)).$$

From the definition of Lebesgue integral and simple computations the following holds true.

Lemma 4.2. *i) Given the expression of $C_\parallel$ and (2), for $\alpha_1 \leq \alpha_2, 0 < c_1 \leq c_2$, if $\varepsilon < e^{-\frac{1}{\alpha_1}}$ then for any Borel functions g s.t. $g\left(\left(\frac{\alpha_1 u}{c_1}\right)^{-\frac{1}{\alpha_1}}, \left(\frac{\alpha_2 u}{c_2}\right)^{-\frac{1}{\alpha_2}}\right)$ is Lebesgue-integrable we have*

$$\int_{0\leq x_1,x_2\leq\varepsilon} g(x_1, x_2)\nu_\parallel(dx_1, dx_2) = \int_{\frac{c_2\varepsilon^{-\alpha_2}}{\alpha_2}}^{+\infty} g\left(\left(\frac{\alpha_1 u}{c_1}\right)^{-\frac{1}{\alpha_1}}, \left(\frac{\alpha_2 u}{c_2}\right)^{-\frac{1}{\alpha_2}}\right) du$$

ii) for $m,\ k \geq 1$ note that $\frac{k}{\alpha_1} + \frac{m}{\alpha_2} - 1 > 0$, and in particular we have

$$\int_{0\leq x_1,x_2\leq\varepsilon} x_1^k x_2^m \nu_\parallel(dx_1, dx_2) = C(k, m)\varepsilon^{m+k\frac{\alpha_2}{\alpha_1}-\alpha_2};$$

iii) for $\ell, k \geq 2$ and $m = 1, 2$ we have:

$$\int_{0<x_m\leq\varepsilon} x_m^k \nu_\perp(dx_1, dx_2) = \int_{0<x_m\leq\varepsilon} x_m^k \nu^{(m)}(dx_m) = C_m(k)\varepsilon^{k-\alpha_m};$$

for $k, m = 1, 2$

$$\int_{0\leq x_1,x_2\leq\varepsilon} x_1^k \nu_\parallel(dx_1, dx_2) = C(k, 0)\varepsilon^{\frac{\alpha_2}{\alpha_1}k-\alpha_2}; \quad \int_{0\leq x_1,x_2\leq\varepsilon} x_2^\ell \nu_\parallel(dx_1, dx_2) = C(0, \ell)\varepsilon^{\ell-\alpha_2};$$

iv) for $m = 1, 2$

$$A_m^\varepsilon \doteq \int_{\varepsilon\leq x_m\leq 1} x_m \nu^{(m)}(dx_m) = c_{A_m}\left[(1-\varepsilon^{1-\alpha_m})I_{\alpha_m\neq 1} + I_{\alpha_m=1}\ln\frac{1}{\varepsilon}\right]. \qquad \square$$

Recall that for $k, m \geq 0$, $C(k, m) \doteq c_2\left(\frac{\alpha_2 c_1}{\alpha_1 c_2}\right)^{\frac{k}{\alpha_1}} \frac{1}{m+\frac{\alpha_2}{\alpha_1}k-\alpha_2} > 0$, and for $k, m = 1, 2$, $C_m(k) = \frac{c_m}{k-\alpha_m}$ and $c_{A_m} \doteq \frac{c_m}{1-\alpha_m} I_{\alpha_m\neq 1} + c_m I_{\alpha_m=1}$.

Note that for $\varepsilon < 1$, $c_{A_m}(1-\varepsilon^{1-\alpha_m}) > 0$ for any $\alpha_m \in]0, 2[$ and that $C(0, m) = \frac{c_2}{m-\alpha_2} = C_2(m)$. The reason why $\int_{0\leq x_1,x_2\leq\varepsilon} x_1^k \nu_\parallel(dx_1, dx_2)$ depends also on α_2 is that the jump sizes of the parallel component of M are connected by $x_2 = f(x_1)$. If $\alpha_1 \leq \alpha_2$ and $0 < c_1 \leq c_2$ then for sufficiently small ε we have $U_1(\varepsilon) \leq U_2(\varepsilon)$, thus $\varepsilon \geq U_1^{-1}(U_2(\varepsilon)) = f^{-1}(\varepsilon)$. It follows that by binding both $x_1 \leq \varepsilon$ and $x_2 = f(x_1) \leq \varepsilon$ we impose that $x_1 \leq f^{-1}(\varepsilon) \wedge \varepsilon = f^{-1}(\varepsilon)$, which is a bound depending on α_2.

Define
$$\tilde{\xi}_i \doteq \frac{\xi_i - E[\xi_1]}{\sqrt{nVar(\xi_1)}}.$$
We know that $\sum_{i=1}^n \tilde{\xi}_i$ is always a tight sequence, since ξ_i are iid and thus $\sqrt{nVar(\xi_1)}$ is the L^2 norm of the centered $\sum_{i=1}^n (\xi_i - E[\xi_i])$. In the next theorem (which is proved in Appendix 2) we compute more explicitly the leading terms of $nE[\xi_1]$ and $\sqrt{nVar(\xi_1)}$.

Theorem 4.3. *Assume* **A2-A5**, $0 < \alpha_1 \leq \alpha_2 < 2, \alpha_2 \geq 1, 0 < c_1 \leq c_2$. *Take* $\varepsilon = h^u$, *any* $u \in]0, \frac{1}{2}[$ *and define*

$$x_\star \doteq \frac{1 + 2u - \sqrt{-4(2\alpha_2 - 1)u^2 + 4u + 1}}{2u} \in (\alpha_2 u, \alpha_2).$$

Then as $\varepsilon \to 0$ *the following quotients are tight:*
i) if $\gamma \in (0,1)$:

$$\frac{\sum_i \xi_i - T(1-\gamma)C(1,1)\varepsilon^{1+\frac{\alpha_2}{\alpha_1}-\alpha_2} I_{\{\alpha_1 > \alpha_2 u\} \cup \{\alpha_1 = \alpha_2 u, \alpha_2 > 1\}} - Thc_{A_1}c_{A_2}F_0(\varepsilon)}{\sqrt{T}\varepsilon^{1-\alpha_2/2}\sqrt{h\varepsilon^{2-\alpha_1}\gamma C_1(2)C(0,2)I_{\{\alpha_1 \leq x_\star\}} + \varepsilon^{2\frac{\alpha_2}{\alpha_1}}(1-\gamma)C(2,2)I_{\{\alpha_1 \geq x_\star\}}}} \tag{13}$$

ii) If $\gamma = 1$:
$$\frac{\sum_i \xi_i - Thc_{A_1}c_{A_2}F_1(\varepsilon)}{\sqrt{T}\sqrt{h}\varepsilon^{2-\alpha_1/2-\alpha_2/2}\sqrt{C_1(2)C_2(2)}}, \tag{14}$$

iii) If $\gamma = 0$: *with* $G \doteq C(2,2) - 2c_{A_1}C(1,2) + c_{A_1}^2 C(0,2)$ *we have*

$$\frac{\sum_i \xi_i - TC(1,1)\varepsilon^{1+\frac{\alpha_2}{\alpha_1}-\alpha_2} I_{\{\alpha_1 > \alpha_2 u\} \cup \{\alpha_1 = \alpha_2 u, \alpha_2 > 1\}} - Thc_{A_1}c_{A_2}F_0(\varepsilon)}{\sqrt{T}\varepsilon^{1-\alpha_2/2}\sqrt{h^2 c_{A_1}^2 C(0,2)I_{\{\alpha_1 < \alpha_2 u\}} + \varepsilon^{2\frac{\alpha_2}{\alpha_1}}\left[C(2,2)I_{\{\alpha_1 > \alpha_2 u\}} + GI_{\{\alpha_1 = \alpha_2 u\}}\right]}}. \tag{15}$$

Remarks on the Theorem statement.

- The term $-4(2\alpha_2 - 1)u^2 + 4u + 1$ within $x_\star$ turns out to be strictly positive for all $u \in (0, \frac{1}{2})$, $\alpha_2 < 2$. Also, for any α_1, α_2 as in the assumptions we have $1 + \frac{\alpha_2}{\alpha_1} - \alpha_2 > 0$.
- The numerator in each quotient is always the difference of $\sum_i \xi_i$ with the leading terms of its (tending to zero) mean. There are parameters choices such that $E[\sum_i \xi_i]$ (or $\sqrt{nVar(\xi_1)}$) has two asymptotically equivalent leading terms.
- As for the denominator in i), the case $\alpha_1 = \alpha_2$ falls within the region $\alpha_1 \geq x_\star$.

Proposition 4.4. *(See the proof in Appendix 2) Assume* $0 < \alpha_1 \le \alpha_2 < 2$, $\alpha_2 \ge 1$, $0 < c_1 \le c_2$, $u \in (0, \frac{1}{2})$. *As* $h \to 0$ *we have* $\frac{\sqrt{nVar(\xi_1)}}{nE[\xi_1]} \to 0$ *in the following cases:*
i) for $\gamma \in [0,1)$*: for any choices of* α_1, α_2 *and* u*, as in the assumptions;*
ii) for $\gamma = 1:$ *on* $\{\alpha_1 < 1, \alpha_2 \ge 1\} \cup \{\alpha_1 = 1 < \alpha_2\}$ *iff* $u \in (\frac{1}{2+\alpha_2-\alpha_1}, \frac{1}{2})$*; on* $\{1 < \alpha_1 \le \alpha_2\}$ *iff* $u \in (\frac{1}{\alpha_1+\alpha_2}, \frac{1}{2})$.
We have $\frac{\sqrt{nVar(\xi_1)}}{nE[\xi_1]} \to +\infty$ *in the following case:*
iii) for $\gamma = 1:$ *on* $\{\alpha_1 = \alpha_2 = 1\}$*, any* $u \in (0, \frac{1}{2})$.

Remark 4.5. *When* $\frac{\sqrt{nVar(\xi_1)}}{nE[\xi_1]} \to 0$ *then the tightness of* $\sum_{i=1}^n \tilde{\xi}_i$ *implies that* $\frac{\sum_{i=1}^n \xi_i}{nE[\xi_1]} \xrightarrow{P} 1$, *that is* $\sum_{i=1}^n \xi_i \sim nE[\xi_1]$. *Otherwise, if* $\frac{\sqrt{nVar(\xi_1)}}{nE[\xi_1]} \to \infty$*, the tightness of* $\sum_{i=1}^n \tilde{\xi}_i$ *only allows us to say that* $\forall \eta > 0\ \exists K_\eta$ *: with probability larger than* $1 - \eta$*, for all sufficiently large* n *we have* $|\sum_{i=1}^n \xi_i| \le \tilde{K}_\eta \sqrt{nVar(\xi_1)}$*, but* $\sum_{i=1}^n \xi_i$ *could tend to 0 faster than* $\sqrt{nVar(\xi_1)}$*. However the following CLT (which is proved in Appendix 2) gives us the exact asymptotic behavior of* $\sum_{i=1}^n \xi_i$.

Theorem 4.6. *When* $\gamma = 1 = \alpha_1 = \alpha_2$*:* $\forall\ u \in (0, \frac{1}{2})$*, with* $\xrightarrow{d}$ *denoting convergence in distribution, we have*

$$\frac{\sum_{i=1}^n \xi_i - nE[\xi_1]}{\sqrt{nVar(\xi_1)}} \xrightarrow{d} \mathcal{N}.$$

Remark. A CLT for $\sum_{i=1}^n \xi_i$ also holds in the case of completely dependent small jumps, i.e. $\gamma = 0$ (see [7], Thm 4.4).

Proof of Theorem 3.1. This is a direct consequence of Theorem 4.3, Proposition 4.4, Remark 4.5 and Theorem 4.6. $\square$

We now proceed to prove Theorem 3.2. Recall that under **A1** we have the property (point iii) within the proof of Theorem 1 in [13]) that a.s.

$$\sup_{1 \le j \le n} \frac{|\Delta_j D^{(m)}|}{\sqrt{2hlog\frac{1}{h}}} \le K_m(\omega) < \infty, \quad m = 1, 2, \tag{16}$$

where $K_m \doteq \sup_{s \in [0,T]} |a|_s + \sup_{s \in [0,T]} |\sigma|_s + 1$ are finite random variables.

By using a localization procedure similar to the one in [9] (sec. 3.6.3) we can assume wlg that the coefficients $a^{(m)}, \sigma^{(m)}, \rho$ in (1) are bounded. In particular, we can take K_m to be constants.

In the following denote, for $m = 1, 2$,

$$N_t^{(m)} = \sum_{s \le t} I_{\{|\Delta X_s^{(m)}| > 1\}}, \ \tilde{N}_t^{(m)} = \sum_{s \le t} I_{\{|\Delta X_s^{(m)}| > \sqrt{r_h}\}}, \ \tilde{V}_t^{(m)} = \sum_{s \le t} I_{\{|\Delta M_s^{(m)}| > \sqrt{r_h}\}}, \ \theta_m = hr_h^{-\frac{\alpha_m}{2}}.$$

K is a mute name for any positive constants: it keeps the same name passing from one side to the other of an inequality/equality, even when the constant changes. For U a rv, we denote $||U||_\ell = E^{\frac{1}{\ell}}[|U|^\ell]$.

Remark 4.7. *1. (Lemma 2 in [1]: note that the expansion (24) and the estimate (50), on which the proof is based, hold for any stable process and any stability index in $(0,2)$, thanks to (2.4.6), (2.4.8) in the cited book of Zolotarev and to the expansion of $p^0(1,x)$ at page 89 in [17]) If $\tilde{L}$ is a symmetric stable process with $\tilde{\tilde{N}}_t = \sum_{s\le t}\Delta\tilde{L}I_{\{|\Delta\tilde{L}_s|>\varepsilon\}}$ and Lévy density $F(dx)=\frac{c}{|x|^{1+\alpha}}dx$, if $\tilde{\theta}=h\varepsilon^{-\alpha}$, then:*

$$P\Big\{\Big|\Delta_i\tilde{L}-\sum_{s\in]t_{i-1},t_i]}\Delta\tilde{L}_sI_{\{|\Delta\tilde{L}_s|>\varepsilon\}}\Big|>\varepsilon\Big\}+P\{|\Delta_i\tilde{L}|>\varepsilon,\Delta_i\tilde{\tilde{N}}=0\}+P\{|\Delta_i\tilde{L}|\le\varepsilon,\Delta_i\tilde{\tilde{N}}=1\}\le K\tilde{\theta}^{\frac{4}{3}}$$

2. ([6], ch.3, Prop. 3.7) For any Lévy process V with Lévy measure ν, then $\sum_{s\le t}I_{\{|\Delta V_s|>\varepsilon\}}$ is a Poisson process with parameter $t\nu\{|x|>\varepsilon\}=tU(\varepsilon)$, where $U(x)$ gives the tail of the jumps sizes measure; it follows that if $\nu(dx)=a|x|^{-1-\alpha}I_{x<0}+bx^{-1-\alpha}I_{x>0}$ with $a,b\ge 0$ and $(a,b)\neq(0,0)$, then with $p\in(0,1)$: $P\{\sum_{s\in]t_{i-1},t_i]}I_{\{|\Delta V_s|>\varepsilon\}}=1\}\sim\tilde{\theta}$,

$$P\{\sum_{s\in]t_{i-1},t_i]}I_{\{|\Delta V_s|>\varepsilon\}}\ge 2\}\sim\tilde{\theta}^2,\ P\{\sum_{s\in]t_{i-1},t_i]}I_{\{|\Delta V_s|\in(\varepsilon(1-p),\varepsilon]\}}=1\}\sim\tilde{\theta}((1-p)^{-\alpha}-1).$$

□

Let us recall that each $M^{(m)}$ is given by the small jumps of a *one-sided* stable process $L^{(m)}$.

Lemma 4.8. *(See the proof in Appendix 2). Let L be a one-sided α-stable process with characteristic triplet $(z,0,c\cdot I_{\{x>0\}}x^{-1-\alpha}dx)$, let $H_1\doteq(L_t-zt)_t$, take $\varepsilon=\varepsilon(h)$ s.t. $h/\varepsilon(h)\to 0$, any constant $p\in(0,1)$ s.t. $p>|z|h/\varepsilon$ and any $q\in(0,1-p)$. For $m=1,2$, $i=1..n$ we have the following.*

1. $P\{\Delta_iN^{(m)}\neq 0,(\Delta_iM^{(m)})^2>r_h\}\le K\frac{h^2}{r_h}$, $P(|\Delta_iM^{(m)}|>K\sqrt{r_h})\le K\theta_m$.
2. $P\{|\Delta_iL|>\varepsilon,\sum_{s\in]t_{i-1},t_i]}I_{\{|\Delta L_s|>\varepsilon\}}=0\}\le K\tilde{\theta}^{4/3}+K\tilde{\theta}(q^{-\alpha}-1)$.
3. $P\{|\Delta_iM^{(m)}|>\sqrt{r_h}(1-p),\Delta_i\tilde{V}^{(m)}=0\}\le K\theta_m^{4/3}+K\theta_m(q^{-\alpha_m}-1)$.
4. $P\{|\Delta_iH_1|\le\varepsilon(1+p),\sum_{s\in]t_{i-1},t_i]}I_{\{|\Delta H_{1s}|>\varepsilon\}}=1\}\le K[\tilde{\theta}^{4/3}+\tilde{\theta}(1-(1+2p)^{-\alpha})]$

   $P\{|\Delta_iL|\le\varepsilon,\sum_{s\in]t_{i-1},t_i]}I_{\{|\Delta L_s|>\varepsilon\}}=1\}\le K[\tilde{\theta}^{4/3}+\tilde{\theta}(1-(1+2p)^{-\alpha})]$.
5. *With* $\varepsilon=\sqrt{r_h}$ *we have* $P\{|\Delta_iM^{(m)}|\le\sqrt{r_h}(1+p),\Delta_i\tilde{V}^{(m)}\ge 1\}\le K\theta_m^{4/3}+K\theta_m(1-(1+2p)^{-\alpha_m})$.

Lemma 4.9. *(See the proof in Appendix 2). Let, for i=1..n, $A_i\subset\Omega$ be independent on $W^{(1)}$ and $W^{(2)}$ and s.t. $\forall i, P(A_i)\le\theta_m$. If each $\sigma^{(j)}$ satisfy (6), then*

i) $\frac{1}{\theta_m}\sum_{i=1}^n\int_{t_{i-1}}^{t_i}\sigma_s^{(1)}dW_s^{(1)}\int_{t_{i-1}}^{t_i}\sigma_s^{(2)}dW_s^{(2)}I_{A_i}\sim\frac{1}{\theta_m}\sum_{i=1}^n\sigma_{t_{i-1}}^{(1)}\Delta_iW^{(1)}\sigma_{t_{i-1}}^{(2)}\Delta_iW^{(2)}I_{A_i}$.

ii) Any $P(A_i)$ is, we have $E[|\sum_{i=1}^n\sigma_{t_{i-1}}^{(1)}\Delta_iW^{(1)}\sigma_{t_{i-1}}^{(2)}\Delta_iW^{(2)}I_{A_i}|]\le KP(A_i)$.

Lemma 4.10. *(See the proof in Appendix 2) With* $\stackrel{ucp}{\to}$ *denoting convergence in probability uniformly on* $[0,T]$ *and* $IC_t \doteq \int_0^t \rho_s \sigma_s^{(1)} \sigma_s^{(2)} ds$, *we have*

$$\frac{1}{\theta_m} \sum_{i=1}^{[t/h]} \int_{t_{i-1}}^{t_i} \sigma_s^{(1)} dW_s^{(1)} \int_{t_{i-1}}^{t_i} \sigma_s^{(2)} dW_s^{(2)} I_{\{\Delta_i \tilde{V}^{(m)} \geq 1\}} \stackrel{ucp}{\to} \frac{c_m}{\alpha_m} IC_t.$$

Lemma 4.11. *(See the proof in Appendix 2) We have*

$$\frac{1}{\theta_1} \sum_{i=1}^{[t/h]} \sigma_{t_{i-1}}^{(1)} \Delta_i W^{(1)} \sigma_{t_{i-1}}^{(2)} \Delta_i W^{(2)} I_{\{\Delta_i \tilde{V}^{(1)} \geq 1, \Delta_i \tilde{V}^{(2)} \geq 1\}} \stackrel{ucp}{\to} (1-\gamma) \frac{c_1}{\alpha_1} IC_t \cdot I_{\{\gamma \in [0,1)\}}.$$

Proof of Theorem 3.2. From now on take a $p \in (0,1)$, h sufficiently small and s.t. $p > \sqrt{h \ln \frac{1}{h}} / \sqrt{r_h}$, $q \in (0, 1-p)$. We can write

$$\hat{IC} - IC = \sum_{k=1}^{4} I_k, \tag{17}$$

where

$$I_1 = \Big[\sum_{i=1}^{n} \Delta_i Y^{(1)} \Delta_i Y^{(2)} I_{\{|\Delta_i Y^{(1)}| \leq 2\sqrt{r_h}\}} I_{\{|\Delta_i Y^{(2)}| \leq 2\sqrt{r_h}\}} - IC \Big],$$

$$I_2 = \sum_{i=1}^{n} \Delta_i Y^{(1)} \Delta_i Y^{(2)} \Big(I_{\{|\Delta_i X^{(1)}| \leq \sqrt{r_h}\}} I_{\{|\Delta_i X^{(2)}| \leq \sqrt{r_h}\}} - I_{\{|\Delta_i Y^{(1)}| \leq 2\sqrt{r_h}\}} I_{\{|\Delta_i Y^{(2)}| \leq 2\sqrt{r_h}\}} \Big),$$

$$I_3 = \sum_{i=1}^{n} (\Delta_i Y^{(1)} \Delta_i M^{(2)} + \Delta_i Y^{(2)} \Delta_i M^{(1)}) I_{\{|\Delta_i X^{(1)}| \leq \sqrt{r_h}\}} I_{\{|\Delta_i X^{(2)}| \leq \sqrt{r_h}\}},$$

$$I_4 = \sum_{i=1}^{n} \Delta_i M^{(1)} \Delta_i M^{(2)} I_{\{|\Delta_i X^{(1)}| \leq \sqrt{r_h}\}} I_{\{|\Delta_i X^{(2)}| \leq \sqrt{r_h}\}}.$$

We know that $I_1/\sqrt{h} \stackrel{st}{\longrightarrow} U$, with U mixed Gaussian rv ([16]). We are now going to show that:

$$I_2 \sim \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \sigma_s^{(1)} dW_s^{(1)} \int_{t_{i-1}}^{t_i} \sigma_s^{(2)} dW_s^{(2)} I_{\{|\Delta_i \tilde{V}^{(2)}| \geq 1\}} \sim \theta_2 = h r_h^{-\frac{\alpha_2}{2}}, \quad I_3 << \sqrt{h}$$

and I_4 is the sum of $\sum_{i=1}^n \xi_i$ with some other terms which however are negligible wrt one of the terms $\sqrt{h}$, θ_2 or $\sum_{i=1}^n \xi_i$. That will prove (8). It then turns out that none of the terms appearing in (8) is always negligible, while depending on the combination of the parameters $\gamma, \alpha_1, \alpha_2$ the leading term is different, and we show (10, 11, 12).

Let us start dealing with $I_2 \doteq I_{2,1} + \mathcal{J}$, where

$$I_{2,1} = \sum_{i=1}^{n} \Delta_i Y^{(1)} \Delta_i Y^{(2)} I_{\{|\Delta_i X^{(1)}| \leq \sqrt{r_h}, |\Delta_i X^{(2)}| \leq \sqrt{r_h}\} \cap \{|\Delta_i Y^{(1)}| \leq 2\sqrt{r_h}, |\Delta_i Y^{(2)}| \leq 2\sqrt{r_h}\}^c},$$

$$\mathcal{J} = -\sum_{i=1}^{n} \Delta_i Y^{(1)} \Delta_i Y^{(2)} \Big(I_{\{|\Delta_i X^{(1)}| \le \sqrt{r_h}, |\Delta_i X^{(2)}| \le \sqrt{r_h}\}^c \cap \{|\Delta_i Y^{(1)}| \le 2\sqrt{r_h}, |\Delta_i Y^{(2)}| \le 2\sqrt{r_h}\}}.$$

We first show that $I_{2,1} << \theta_2$. In fact, for each i, on the set highlighted by the indicator we have $|\Delta_i Y^{(m)}| > 2\sqrt{r_h}$ for at least one $m \in \{1,2\}$, and, using (16), we have $|\Delta_i J^{(m)}| + K\sqrt{h \ln \frac{1}{h}} \ge |\Delta_i D^{(m)} + \Delta_i J^{(m)}| = |\Delta_i Y^{(m)}| > 2\sqrt{r_h}$, which implies that $|\Delta_i J^{(m)}| \ge 2\sqrt{r_h}(1-p)$, thus $|\Delta_i J^{(m)}| \ne 0$. However $|\Delta_i X^{(m)}| \le \sqrt{r_h}$, and so $|\Delta_i J^{(m)} + \Delta_i M^{(m)}| - |\Delta_i D^{(m)}| < |\Delta_i X^{(m)}| \le \sqrt{r_h}$ implies on one hand that $|\Delta_i J^{(m)} + \Delta_i M^{(m)}| < \sqrt{r_h}(1+p)$, and on the other hand that, considering a sufficiently small h, that $1 - |\Delta_i M^{(m)}| < |\Delta_i J^{(m)}| - |\Delta_i M^{(m)}| < |\Delta_i J^{(m)} + \Delta_i M^{(m)}| < \sqrt{r_h}(1+p)$, and thus, for sufficiently small h, $|\Delta_i M^{(m)}| > 1 - \sqrt{r_h}(1+p) > \sqrt{r_h}$. It follows that $\forall i = 1..n$ there is an index m_i s.t. $\{|\Delta_i X^{(1)}| \le \sqrt{r_h}, |\Delta_i X^{(2)}| \le \sqrt{r_h}\} \cap \{|\Delta_i Y^{(1)}| \le 2\sqrt{r_h}, |\Delta_i Y^{(2)}| \le 2\sqrt{r_h}\}^c \subset \{\Delta_i N^{(m_i)} \ne 0, \Delta_i M^{(m_i)} > \sqrt{r_h}\}$, thus, using Lemma 4.8 point 1, $P\left\{\frac{I_{2,1}}{\theta_2} \ne 0\right\} \le \sum_{i=1}^n P\{ \Delta_i N^{(m_i)} \ne 0, \Delta_i M^{(m_i)} > \sqrt{r_h}\} \le K\frac{h}{r_h} \to 0$, which implies that $\frac{I_{2,1}}{\theta_2} \xrightarrow{P} 0$.

As for term $\mathcal{J}$, on $\{|\Delta_i Y^{(m)}| \le 2\sqrt{r_h}\}$ we have $|\Delta_i J^{(m)}| - |\Delta_i D^{(m)}| < |\Delta_i Y^{(m)}| \le 2\sqrt{r_h}$ and thus $|\Delta_i J^{(m)}| < 2\sqrt{r_h}(1+p) < 1$, which implies that $\Delta_i J^{(m)} = 0$, i.e. $\Delta_i Y^{(m)} = \Delta_i D^{(m)}$. Thus, calling

$$\mathcal{B}_i = \{|\Delta_i X^{(1)}| \le \sqrt{r_h}, |\Delta_i X^{(2)}| \le \sqrt{r_h}\}^c \cap \{|\Delta_i Y^{(1)}| \le 2\sqrt{r_h}, |\Delta_i Y^{(2)}| \le 2\sqrt{r_h}\},$$

we have $\mathcal{J} = \sum_{i=1}^n \Delta_i Y^{(1)} \Delta_i Y^{(2)} I_{\mathcal{B}_i} \doteq \sum_{k=2}^4 I_{2,k}$, where

$$I_{2,2} = \sum_{i=1}^n \int_{t_{i-1}}^{t_i} a_s^{(1)} ds \int_{t_{i-1}}^{t_i} a_s^{(2)} ds \; I_{\mathcal{B}_i}, \quad I_{2,4} = \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \sigma_s^{(1)} dW_s^{(1)} \int_{t_{i-1}}^{t_i} \sigma_s^{(2)} dW_s^{(2)} I_{\mathcal{B}_i},$$

$$I_{2,3} = \sum_{i=1}^n \Big(\int_{t_{i-1}}^{t_i} a_s^{(2)} ds \int_{t_{i-1}}^{t_i} \sigma_s^{(1)} dW_s^{(1)} + \int_{t_{i-1}}^{t_i} a_s^{(1)} ds \int_{t_{i-1}}^{t_i} \sigma_s^{(2)} dW_s^{(2)} \Big) I_{\mathcal{B}_i}.$$

We show that $I_{2,4}$ is the leading term and it asymptotically behaves as θ_2. As for $I_{2,2}$, by the boundedness of each $a^{(m)}$ we have $E[\frac{|I_{2,2}|}{\theta_2}] \le \frac{h}{\theta_2} \to 0$.

As for $I_{2,3}$, note that on $\{|\Delta_i X^{(m)}| > \sqrt{r_h}, |\Delta_i J^{(m)}| = 0\}$ we have $|\Delta_i M^{(m)}| + K\sqrt{h \ln \frac{1}{h}} > |\Delta_i M^{(m)}| + |\Delta_i D^{(m)}| \ge |\Delta_i D^{(m)} + \Delta_i M^{(m)}| = |\Delta_i X^{(m)}| > \sqrt{r_h}$ thus $|\Delta_i M^{(m)}| > \sqrt{r_h} - K\sqrt{h \ln \frac{1}{h}} > \sqrt{r_h}(1-p)$. Using also Lemma 4.8 point 1 and noting that $\theta_1 \le \theta_2$, it follows that

$$\frac{E[|I_{2,3}|]}{\theta_2} \le \frac{K}{\theta_2} \sum_{i=1}^n h\sqrt{h \ln \frac{1}{h}} \Big(P\{|\Delta_i M^{(1)}| > K\sqrt{r_h}\} + P\{|\Delta_i M^{(2)}| > K\sqrt{r_h}\} \Big) \le K\sqrt{h \ln \frac{1}{h}},$$

which tends to 0.

As for $I_{2,4}$, firstly we show that

$$\frac{I_{2,4}}{\theta_2} \sim \frac{1}{\theta_2} \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \sigma_s^{(1)} dW_s^{(1)} \int_{t_{i-1}}^{t_i} \sigma_s^{(2)} dW_s^{(2)} \Big(I_{\{|\Delta_i X^{(1)}| > \sqrt{r_h}\}} + I_{\{|\Delta_i X^{(2)}| > \sqrt{r_h}\}} \Big) - B_n, \quad (18)$$

where $B_n \sim \theta_1/\theta_2(1-\gamma)$. Let us begin showing that

$$\frac{I_{2,4}}{\theta_2} \sim \frac{1}{\theta_2}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\sigma_s^{(1)}dW_s^{(1)}\int_{t_{i-1}}^{t_i}\sigma_s^{(2)}dW_s^{(2)} I_{\{|\Delta_i X^{(1)}|\le\sqrt{r_h},|\Delta_i X^{(2)}|\le\sqrt{r_h}\}^c} : \tag{19}$$

as argued just after the expression of $I_{2,1}$, $\{|\Delta_i Y^{(m)}| > 2\sqrt{r_h}\} \subset \{\Delta_i N^{(m)} \neq 0\}$, thus

$$\frac{1}{\theta_2}\sum_{i=1}^{n}\left|\int_{t_{i-1}}^{t_i}\sigma_s^{(1)}dW_s^{(1)}\int_{t_{i-1}}^{t_i}\sigma_s^{(2)}dW_s^{(2)}\right| I_{\{|\Delta_i X^{(m)}|\le\sqrt{r_h},m=1,2\}^c\cap\{|\Delta_i Y^{(m)}|\le 2\sqrt{r_h},m=1,2\}^c} \le$$

$$\frac{1}{\theta_2}\sum_{i=1}^{n}\left|\int_{t_{i-1}}^{t_i}\sigma_s^{(1)}dW_s^{(1)}\int_{t_{i-1}}^{t_i}\sigma_s^{(2)}dW_s^{(2)}\right| \left(I_{\{\Delta_i N^{(1)}\neq 0\}} + I_{\{\Delta_i N^{(2)}\neq 0\}}\right) : \tag{20}$$

$$\frac{\sum_{i=1}^{n}\left|\int_{t_{i-1}}^{t_i}\sigma_s^{(1)}dW_s^{(1)}\int_{t_{i-1}}^{t_i}\sigma_s^{(2)}dW_s^{(2)}\right| I_{\{\Delta_i N^{(m)}\neq 0\}}}{\theta_2} \le \frac{K}{\theta_2}\sum_{i=1}^{n} h\ln\frac{1}{h} I_{\{\Delta_i N^{(m)}\neq 0\}}$$

has expectation bounded by $K\frac{h\ln\frac{1}{h}}{\theta_2} = \varepsilon^{\alpha_2}\ln\frac{1}{h} \to 0$, thus (19) follows. Since now $I_{(A\cap B)^c} = I_{A^c} + I_{B^c} - I_{A^c\cap B^c}$, (19) coincides with

$$\frac{1}{\theta_2}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\sigma_s^{(1)}dW_s^{(1)}\int_{t_{i-1}}^{t_i}\sigma_s^{(2)}dW_s^{(2)}\Big[I_{\{|\Delta_i X^{(1)}|>\sqrt{r_h}} + I_{\{|\Delta_i X^{(2)}|>\sqrt{r_h}\}} - I_{\{|\Delta_i X^{(1)}|>\sqrt{r_h},|\Delta_i X^{(2)}|>\sqrt{r_h}\}}\Big]. \tag{21}$$

We now show that

$$B_n \doteq \frac{1}{\theta_2}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\sigma_s^{(1)}dW_s^{(1)}\int_{t_{i-1}}^{t_i}\sigma_s^{(2)}dW_s^{(2)} I_{\{|\Delta_i X^{(1)}|>\sqrt{r_h},|\Delta_i X^{(2)}|>\sqrt{r_h}\}} \sim \frac{\theta_1}{\theta_2}(1-\gamma) : \tag{22}$$

the left term is asymptotically equivalent to

$$\frac{1}{\theta_2}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\sigma_s^{(1)}dW_s^{(1)}\int_{t_{i-1}}^{t_i}\sigma_s^{(2)}dW_s^{(2)} I_{\{|\Delta_i\tilde V^{(1)}|\ge 1,|\Delta_i\tilde V^{(2)}|\ge 1\}}] \tag{23}$$

because

$$\frac{1}{\theta_2}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\sigma_s^{(1)}dW_s^{(1)}\int_{t_{i-1}}^{t_i}\sigma_s^{(2)}dW_s^{(2)}\left(I_{\{|\Delta_i X^{(1)}|>\sqrt{r_h},|\Delta_i X^{(2)}|>\sqrt{r_h}\}} - I_{\{\Delta_i\tilde V^{(1)}\ge 1,\Delta_i\tilde V^{(2)}\ge 1\}}\right) =$$

$$\frac{1}{\theta_2}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\sigma_s^{(1)}dW_s^{(1)}\int_{t_{i-1}}^{t_i}\sigma_s^{(2)}dW_s^{(2)}\Big(I_{\{|\Delta_i X^{(m)}|>\sqrt{r_h},m=1,2,\text{but }\Delta_i\tilde V^{(\ell)}=0\text{for at least one index }\ell_i\}} \tag{24}$$

$$-I_{\{\Delta_i\tilde V^{(m)}\ge 1,m=1,2,\text{but }|\Delta_i X^{(\ell)}|\le\sqrt{r_h}\text{for at least one index }\ell_i\}}\Big).$$

On $\{|\Delta_i X^{(\ell_i)}| > \sqrt{r_h}, \Delta_i\tilde V^{(\ell_i)} = 0\}$ either $\Delta_i J^{(\ell_i)} \neq 0$ or $\Delta_i J^{(\ell_i)} = 0$. In this last case, as above (18), $|\Delta_i M^{(\ell_i)}| > \sqrt{r_h}(1-p)$; further also on $\{\Delta_i\tilde V^{(\ell_i)} \ge 1, |\Delta_i X^{(\ell_i)}| \le \sqrt{r_h}\}$, either

$\Delta_i J^{(\ell_i)} \neq 0$ or $\Delta_i J^{(\ell_i)} = 0$, and in this last case we have $|\Delta_i M^{(\ell_i)}| = |\Delta_i X^{(\ell_i)} - \Delta_i D^{(\ell_i)}| \leq |\Delta_i X^{(\ell_i)}| + |\Delta_i D^{(\ell_i)}| \leq \sqrt{r_h}(1 + \sqrt{h \ln \frac{1}{h}}/\sqrt{r_h}) < \sqrt{r_h}(1 + h^\eta)$, with $0 < \eta < 1/2 - u$. Thus the factors within brackets in (24) are

$$I_{\{|\Delta_i M^{(\ell_i)}|>\sqrt{r_h}(1-p),\Delta_i \tilde{V}^{(\ell_i)}=0,\Delta_i J^{(\ell_i)}=0\}} - I_{\{|\Delta_i X^{(\ell_i)}|\leq\sqrt{r_h},\Delta_i \tilde{V}^{(\ell_i)}=0,\Delta_i J^{(\ell_i)}=0,|\Delta_i M^{(\ell_i)}|>\sqrt{r_h}(1-p)\}} + I_{\{|\Delta_i X^{(\ell_i)}|>\sqrt{r_h},\Delta_i \tilde{V}^{(\ell_i)}=0,\Delta_i J^{(\ell_i)}\neq0\}} + I_{\{|\Delta_i M^{(\ell_i)}|\leq\sqrt{r_h}(1+h^\eta),\Delta_i \tilde{V}^{(\ell_i)}\geq1,\Delta_i J^{(\ell_i)}=0\}} - I_{\{|\Delta_i X^{(\ell_i)}|>\sqrt{r_h},|\Delta_i M^{(\ell_i)}|\leq\sqrt{r_h}(1+h^\eta),\Delta_i \tilde{V}^{(\ell_i)}\geq1,\Delta_i J^{(\ell_i)}=0\}} + I_{\{|\Delta_i X^{(\ell_i)}|\leq\sqrt{r_h},\Delta_i \tilde{V}^{(\ell_i)}\geq1,\Delta_i J^{(\ell_i)}\neq0\}}. \tag{25}$$

Firstly, as for (20), the third and sixth terms are negligible. As for the fifth term, since $\{|\Delta_i X^{(\ell_i)}| > \sqrt{r_h}\}$ and $\Delta_i J^{(\ell_i)} = 0$, we have $\Delta_i M^{(\ell_i)} > \sqrt{r_h}(1-p)$, which leads to $\sqrt{r_h}(1-p) < |\Delta_i M^{(\ell_i)}| \leq \sqrt{r_h}(1+h^\eta)$. However, by Lemma 6 in [1], we have

$$P\{(1-p)h^u < |\Delta_i M^{(m)}| \leq h^u(1+h^\eta)\} \leq K h^{1-\alpha_m u+\phi}, \quad \phi \doteq \eta \wedge \alpha_m u \wedge (1-\alpha_m u - 2\eta) > 0.$$

Applying the Hölder inequality with conjugate exponents $s_1, s_2 > 1$ to the fifth term we reach

$$\frac{1}{\theta_2}\sum_{i=1}^n E\left[\left|\int_{t_{i-1}}^{t_i} \sigma_s^{(1)} dW_s^{(1)} \int_{t_{i-1}}^{t_i} \sigma_s^{(2)} dW_s^{(2)}\right| I_{\{|\Delta_i X^{(\ell_i)}|>\sqrt{r_h},|\Delta_i M^{(\ell_i)}|\leq\sqrt{r_h}(1+h^\eta),\Delta_i \tilde{V}^{(\ell_i)}\geq1,\Delta_i J^{(\ell_i)}=0\}}\right]$$

$$\leq K\sum_{i=1}^n h \frac{P^{\frac{1}{s_2}}\{\sqrt{r_h}(1-p) < |\Delta_i M^{(\ell_i)}| \leq \sqrt{r_h}(1+h^\eta)\}}{\theta_2} \leq K h^{(1-\alpha_2 u+\phi)\frac{1}{s_2} - (1-\alpha_2 u)},$$

which, for s_2 properly chosen close to 1, tends to 0, since $(1-\alpha_2 u+\phi) > 1-\alpha_2 u$.

As for the second term in (25), we have that on $\{|\Delta_i X^{(\ell_i)}| \leq \sqrt{r_h}, |\Delta_i M^{(\ell_i)}| > \sqrt{r_h}(1-p)\}$ either $|\Delta_i M^{(\ell_i)}| > \sqrt{r_h}(1+h^\eta)$, which leads to $\Delta_i J^{(\ell_i)} \neq 0$ and thus to a negligible term, or $\sqrt{r_h}(1-p) < |\Delta_i M^{(\ell_i)}| \leq \sqrt{r_h}(1+h^\eta)$, which also leads, by the same reasoning as just above, to a negligible term.

Finally, using again the negligibility of $\frac{1}{\theta_2}\sum_{i=1}^n |\int_{t_{i-1}}^{t_i} \sigma_s^{(1)} dW_s^{(1)} \int_{t_{i-1}}^{t_i} \sigma_s^{(2)} dW_s^{(2)}| I_{\{\Delta_i J^{(\ell_i)}\neq0\}}$, as for the first and fourth terms in (25) we have

$$\sum_{i=1}^n \frac{\int_{t_{i-1}}^{t_i}\sigma_s^{(1)} dW_s^{(1)} \int_{t_{i-1}}^{t_i}\sigma_s^{(2)} dW_s^{(2)}}{\theta_2}\Big(I_{\{|\Delta_i M^{(\ell_i)}|>\sqrt{r_h}(1-p),\Delta_i \tilde{V}^{(\ell_i)}=0\}\cup\{|\Delta_i M^{(\ell_i)}|\leq\sqrt{r_h}(1+h^\eta),\Delta_i \tilde{V}^{(\ell_i)}\geq1\}}\Big) I_{\{\Delta_i J^{(\ell_i)}=0\}}$$

$$\sim \sum_{i=1}^n \frac{\int_{t_{i-1}}^{t_i}\sigma_s^{(1)} dW_s^{(1)}}{\theta_2}\int_{t_{i-1}}^{t_i}\sigma_s^{(2)} dW_s^{(2)}\Big(I_{\{|\Delta_i M^{(\ell_i)}|>\sqrt{r_h}(1-p),\Delta_i \tilde{V}^{(\ell_i)}=0\}\cup\{\Delta_i \tilde{V}^{(\ell_i)}\geq1,|\Delta_i M^{(\ell_i)}|\leq\sqrt{r_h}(1+h^\eta)\}}\Big)$$

$$\sim \frac{1}{\theta_2}\sum_{i=1}^n \sigma_{i-1}^{(1)}\Delta_i W^{(1)} \sigma_{i-1}^{(2)}\Delta_i W^{(2)}\Big(I_{\{|\Delta_i M^{(\ell_i)}|>\sqrt{r_h}(1-p),\Delta_i \tilde{V}^{(\ell_i)}=0\}\cup\{\Delta_i \tilde{V}^{(\ell_i)}\geq1,|\Delta_i M^{(\ell_i)}|\leq\sqrt{r_h}(1+h^\eta)}\Big),$$

and writing $E[\sum_{i=1}^n |\eta_i|] = E[\sum_{i=1}^n E_{i-1}|\eta_i|]$, using the independence, the Hölder inequality for $E[|\sigma_{i-1}^{(1)}\ \Delta_i W^{(1)} \sigma_{i-1}^{(2)}\Delta_i W^{(2)}|]$, Lemma 4.8 points 3 and 5, and recalling that $\theta_1 \leq \theta_2$, then $\forall p \in (0,1),\ p > h^\eta > \sqrt{h \ln \frac{1}{h}}/\sqrt{r_h}, \forall q \in (0, 1-p)$ the expectation of the sum of the absolute values of the terms in the previous display is dominated by

$$\frac{Kh}{\theta_2} \sum_{i=1}^{n} \Big(P\{|\Delta_i M^{(\ell_i)}| > \sqrt{r_h}(1-p), \Delta_i \tilde{V}^{\ell_i} = 0\} + P\{\Delta_i \tilde{V}^{\ell_i} \geq 1, |\Delta_i M^{(\ell_i)}| \leq \sqrt{r_h}(1+p)\}\Big)$$

$$\leq K \frac{(\theta_2^{\frac{4}{3}} + \theta_2(q^{-\alpha_2} - 1) + \theta_2(1-(1+2p)^{-\alpha_2}))}{\theta_2} \to K(q^{-\alpha_2} - (1+2p)^{-\alpha_2}).$$

However if we take $p \to 0$ and $q \to 1$ we reach that the limit in probability of (24) is 0 and (23) is asymptotically equivalent to (22).

Now, by Lemma 4.9 and Lemma 4.11, (23) has the same rate as

$$\frac{1}{\theta_2} \sum_{i=1}^{n} \sigma_{t_{i-1}}^{(1)} \Delta_i W^{(1)} \sigma_{t_{i-1}}^{(2)} \Delta_i W^{(2)} I_{\{|\Delta_i \tilde{V}^{(1)}| \geq 1, |\Delta_i \tilde{V}^{(2)}| \geq 1\}}] \sim \frac{\theta_1}{\theta_2}(1-\gamma),$$

so (22) is shown, and, by (21), also (18) is true.

Now, by reasoning exactly as for (23) we have

$$\frac{1}{\theta_2} \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \sigma_s^{(1)} dW_s^{(1)} \int_{t_{i-1}}^{t_i} \sigma_s^{(2)} dW_s^{(2)} I_{\{|\Delta_i X^{(m)}| > \sqrt{r_h}\}} \sim$$

$$\frac{1}{\theta_2} \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \sigma_s^{(1)} dW_s^{(1)} \int_{t_{i-1}}^{t_i} \sigma_s^{(2)} dW_s^{(2)} I_{\{\Delta_i \tilde{V}^{(m)} \geq 1\}},$$

which, by Lemma 4.10, is asymptotically equivalent to $\frac{\theta_m}{\theta_2}$. In particular

$$\frac{I_{2,4}}{\theta_2} \approx IC\Big[\frac{\theta_1}{\theta_2}\frac{c_1}{\alpha_1} + \frac{c_2}{\alpha_2} - \frac{\theta_1}{\theta_2}(1-\gamma)\frac{c_1}{\alpha_1}\Big]:$$

if $\alpha_1 < \alpha_2$ we deduce that $I_2 \sim I_{2,4} \sim \theta_2$, while if $\alpha_1 = \alpha_2 = \alpha$, then $I_{2,4}/\theta_2 \approx c_1/\alpha + c_2/\alpha - (1-\gamma)c_1/\alpha = (c_2 + \gamma c_1)/\alpha$, which is always non zero because $\gamma \geq 0$ and $c_m > 0$.

We now show that I_3 in (17) is negligible wrt $\sqrt{h}$. Here we adjust to the bivariate case the proof given in [5] for the univariate case. $I_3/\sqrt{h}$ is the sum of two terms of type $\frac{1}{\sqrt{h}} \sum_{i=1}^n \Delta_i Y^{(m)} \Delta_i M^{(\ell)} I_{\{|\Delta_i X^{(1)}| \leq \sqrt{r_h}\}}\ I_{\{|\Delta_i X^{(2)}| \leq \sqrt{r_h}\}}$ with $(m,\ell) \in \{(1,2),(2,1)\}$, that we can treat at the same time. The last expression equals

$$\sum_{i=1}^{n} \frac{\Delta_i D^{(m)} \Delta_i M^{(\ell)}}{\sqrt{h}} I_{\{|\Delta_i X^{(1)}| \leq \sqrt{r_h}, |\Delta_i X^{(2)}| \leq \sqrt{r_h}\}} + \sum_{i=1}^{n} \frac{\Delta_i J^{(m)} \Delta_i M^{(\ell)}}{\sqrt{h}} I_{\{|\Delta_i X^{(1)}| \leq \sqrt{r_h}, |\Delta_i X^{(2)}| \leq \sqrt{r_h}\}}. \tag{26}$$

As for the second term, as already commented just after the definition of $I_{2,1}$, on $\{|\Delta_i X^{(m)}| \leq \sqrt{r_h}, \Delta_i J^{(m)} \neq 0\}$ we have $\{|\Delta_i M^{(m)}| > \sqrt{r_h}\}$, thus, by Lemma 4.8 point 1,

$$P\Big\{\frac{1}{\sqrt{h}} \sum_{i=1}^{n} \Delta_i J^{(m)} \Delta_i M^{(\ell)} I_{\{|\Delta_i X^{(1)}| \leq \sqrt{r_h}, |\Delta_i X^{(2)}| \leq \sqrt{r_h}\}} \neq 0\Big\} \leq \sum_{i=1}^{n} \{\Delta_i J^{(m)} \neq 0, |\Delta_i M^{(m)}| > \sqrt{r_h}\},$$

which tends to 0, thus the second term of (26) tends to 0 in probability.

As for the first term, on $\{|\Delta_i X^{(\ell)}| \leq \sqrt{r_h}\}$ we have $|\Delta_i X^{(\ell)}| > |\Delta_i Z^{(\ell)}| - |\Delta_i D^{(\ell)}|$ then $|\Delta_i Z^{(\ell)}| < |\Delta_i X^{(\ell)}| + |\Delta_i D^{(\ell)}| \leq \sqrt{r_h} + \sqrt{h \ln \frac{1}{h}} \leq 2\sqrt{r_h}$, thus the first term is dominated by

$$\frac{1}{\sqrt{h}} \sum_{i=1}^{n} \Delta_i D^{(m)} \Delta_i M^{(\ell)} I_{\{|\Delta_i X^{(1)}| \leq \sqrt{r_h}, |\Delta_i X^{(2)}| \leq \sqrt{r_h}, |\Delta_i Z^{(\ell)}| \leq 2\sqrt{r_h}\}} :$$

since, similarly as above, the terms where $\Delta_i J^{(\ell)} \neq 0$ are negligible, and $\{|\Delta_i Z^{(\ell)}| \leq 2\sqrt{r_h}, \Delta_i J^{(\ell)} = 0\} = \{|\Delta_i M^{(\ell)}| \leq 2\sqrt{r_h}, \Delta_i J^{(\ell)} = 0\}$, we are left with $\frac{1}{\sqrt{h}} \sum_{i=1}^{n} \Delta_i D^{(m)} \cdot$ $\cdot \Delta_i M^{(\ell)} I_{\{|\Delta_i X^{(1)}| \leq \sqrt{r_h}, |\Delta_i X^{(2)}| \leq \sqrt{r_h}, |\Delta_i M^{(\ell)}| \leq 2\sqrt{r_h}, \Delta_i J^{(\ell)} = 0\}}$. However again the same sum above with $\{\Delta_i J^{(\ell)} \neq 0\}$ in place of $\{\Delta_i J^{(\ell)} = 0\}$ is negligible, because on $\{|\Delta_i X^{(\ell)}| \leq \sqrt{r_h}, |\Delta_i M^{(\ell)}| \leq 2\sqrt{r_h}, \Delta_i J^{(\ell)} \neq 0\}$ we still have $|\Delta_i M^{(\ell)}| > \sqrt{r_h}$. So we remain with

$$\frac{1}{\sqrt{h}} \sum_{i=1}^{n} \Delta_i D^{(m)} \Delta_i M^{(\ell)} I_{\{|\Delta_i X^{(1)}| \leq \sqrt{r_h}, |\Delta_i X^{(2)}| \leq \sqrt{r_h}, |\Delta_i M^{(\ell)}| \leq 2\sqrt{r_h}\}}. \tag{27}$$

Now, by Lemma 3.1 in [5] we know that on $|\Delta_i M^{(\ell)}| \leq 2\sqrt{r_h}$ we have $\Delta_i M^{(\ell)} = \Delta_i M^{(\ell)h} - h \int_{2v_h}^{1} x \nu^{(\ell)}(dx)$, where $\Delta_i M^{(\ell)h} = \int_{t_{i-1}}^{t_i} \int_{0 < x \leq 2v_h} x \tilde{\mu}^{\ell}(dx, ds)$, and v_h is a given sequence satisfying $0 < v_h \leq r_h^{1/4}$. As a consequence, exactly as in (43) of [5], the component

$$\frac{1}{\sqrt{h}} \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} a_s^{(m)} ds \Delta_i M^{(\ell)} I_{\{|\Delta_i X^{(1)}| \leq \sqrt{r_h}, |\Delta_i X^{(2)}| \leq \sqrt{r_h}, |\Delta_i M^{(\ell)}| \leq 2\sqrt{r_h}\}}$$

of (27) tends to zero in probability. Now we show the negligibility of

$$\frac{1}{\sqrt{h}} \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \sigma_s^{(m)} dW_s^{(m)} \Delta_i M^{(\ell)h} I_{\{|\Delta_i X^{(1)}| \leq \sqrt{r_h}, |\Delta_i X^{(2)}| \leq \sqrt{r_h}, |\Delta_i M^{(\ell)}| \leq 2\sqrt{r_h}\}} :$$

in fact, by the independence of $W^{(m)}$ on $\tilde{\mu}^{(\ell)}$ also $[\int_0^{\cdot} \sigma_s^{(m)} dW_s^{(m)}, M^{(\ell)h}] \equiv 0$, and the squared norm $||.||_2^2$ of the last display is dominated by

$$\frac{1}{h} E\Big[\Big(\sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \sigma_s^{(m)} dW_s^{(m)} \Delta_i M^{(\ell)h}\Big)^2\Big] = \frac{1}{h} \sum_{i=1}^{n} E\Big[\Big(\int_{t_{i-1}}^{t_i} \sigma_s^{(m)} dW_s^{(m)}\Big)^2 \Big(\Delta_i M^{(\ell)h}\Big)^2\Big]$$

$$\leq \frac{K}{h} n \cdot h \ln\Big(\frac{1}{h}\Big) \cdot h \int_0^{r_h^{1/4}} x^2 \nu^{(\ell)}(dx) \leq K r_h^{\frac{2-\alpha_\ell}{4}} \log \frac{1}{h} \to 0.$$

Finally we show the negligibility also of

$$\frac{1}{\sqrt{h}} \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \sigma_s^{(m)} dW_s^{(m)} h \int_{2v_h}^{1} x \nu^{(\ell)}(dx) I_{\{|\Delta_i X^{(1)}| \leq \sqrt{r_h}, |\Delta_i X^{(2)}| \leq \sqrt{r_h}, |\Delta_i M^{(\ell)}| \leq 2\sqrt{r_h}\}} :$$

in fact recall that $\int_{2v_h}^1 x\nu^{(\ell)}(dx) = c_{A_\ell}\Big[(1-(2v_h)^{1-\alpha_\ell})I_{\alpha_\ell\neq 1} + \ln\frac{1}{2v_h}I_{\alpha_\ell=1}\Big]$ is positive for all the values of $\alpha_\ell \in (0,2)$, so the norm $||.||_1$ of the last display is dominated by

$$\sqrt{h}c_{A_\ell}\Big[(1-(2v_h)^{1-\alpha_\ell})I_{\alpha_\ell\neq 1} + \ln\frac{1}{2v_h}I_{\alpha_\ell=1}\Big]E\Big[\Big|\sum_{i=1}^n\int_{t_{i-1}}^{t_i}\sigma_s^{(m)}dW_s^{(m)}\Big|\Big] \tag{28}$$

and noting that if $i\neq j$ then $E[\int_{t_{i-1}}^{t_i}\sigma_s^{(m)}dW_s^{(m)}\int_{t_{j-1}}^{t_j}\sigma_s^{(m)}dW_s^{(m)}] = E[\int \sigma_s^{(m)}I_{s\in]t_{i-1},t_i]}\sigma_s^{(m)}\cdot$ $\cdot I_{s\in]t_{j-1},t_j]}ds] = 0$, and that

$$E\Big[\Big|\sum_{i=1}^n\int_{t_{i-1}}^{t_i}\sigma_s^{(m)}dW_s^{(m)}\Big|\Big] \le ||\sum_{i=1}^n\int_{t_{i-1}}^{t_i}\sigma_s^{(m)}dW_s^{(m)}||_2 = \sqrt{E[\sum_{i=1}^n(\int_{t_{i-1}}^{t_i}\sigma_s^{(m)}dW_s^{(m)})^2]} = O(1).$$

It follows that (28) is dominated by $K\sqrt{h}\Big[|1-(2v_h)^{1-\alpha_\ell}|I_{\alpha_\ell\neq 1} + \ln\frac{1}{2v_h}I_{\alpha_\ell=1}\Big] \to 0$.

We now deal with I_4 of (17). We have

$$I_4 = \sum_{i=1}^n \Delta_i M^{(1)}\Delta_i M^{(2)} I_{\{|\Delta_i X^{(1)}|\le\sqrt{r_h}, |\Delta_i X^{(2)}|\le\sqrt{r_h}\}}$$

$$= \sum_{i=1}^n \Delta_i M^{(1)}\Delta_i M^{(2)}\Big[I_{\{\Delta_i\tilde N^{(1)}=0,\Delta_i\tilde N^{(2)}=0\}} + I_{\{\Delta_i\tilde N^{(1)}=0,\Delta_i\tilde N^{(2)}=0\}^c}\Big] I_{\{|\Delta_i X^{(1)}|\le\sqrt{r_h}, |\Delta_i X^{(2)}|\le\sqrt{r_h}\}}$$

$$= \sum_{i=1}^n \Delta_i M^{(1)}\Delta_i M^{(2)}\Big[I_{\{\Delta_i\tilde N^{(1)}=0,\Delta_i\tilde N^{(2)}=0\}} - I_{\{\Delta_i\tilde N^{(1)}=0,\Delta_i\tilde N^{(2)}=0\}\cap\{|\Delta_i X^{(1)}|\le\sqrt{r_h}, |\Delta_i X^{(2)}|\le\sqrt{r_h}\}^c} +$$

$$I_{\{\Delta_i\tilde N^{(1)}=0,\Delta_i\tilde N^{(2)}=0\}^c\cap\{|\Delta_i X^{(1)}|\le\sqrt{r_h}, |\Delta_i X^{(2)}|\le\sqrt{r_h}\}}\Big].$$

However, where both $\Delta_i\tilde N^{(1)} = 0, \Delta_i\tilde N^{(2)} = 0$, we have $\Delta_i M^{(1)}\Delta_i M^{(2)} = \xi_i$, thus $I_4 = \sum_{k=1}^4 I_{4,k}$, where

$$I_{4,2} = -\sum_{i=1}^n \xi_i I_{\{\Delta_i\tilde N^{(1)}=0,\Delta_i\tilde N^{(2)}=0\}^c},\ I_{4,3} = -\sum_{i=1}^n \xi_i I_{\{\Delta_i\tilde N^{(1)}=0,\Delta_i\tilde N^{(2)}=0\}\cap\{|\Delta_i X^{(1)}|\le\sqrt{r_h}, |\Delta_i X^{(2)}|\le\sqrt{r_h}\}^c}$$

$$I_{4,1} = \sum_{i=1}^n \xi_i, \quad I_{4,4} = \sum_{i=1}^n \Delta_i M^{(1)}\Delta_i M^{(2)} I_{\{\Delta_i\tilde N^{(1)}=0,\Delta_i\tilde N^{(2)}=0\}^c\cap\{|\Delta_i X^{(1)}|\le\sqrt{r_h}, |\Delta_i X^{(2)}|\le\sqrt{r_h}\}}.$$

We are going to show that the terms $I_{4,2}, I_{4,4}$ are negligible wrt θ_2, while $I_{4,3}$ is negligible either wrt θ_2 or wrt $\sum_{i=1}^n \xi_i$, depending on the parameters values. As for $I_{4,2}$, using again that $I_{A\cup B} = I_A + I_B - I_{A\cap B}$, it is sufficient to show that both $\sum_{i=1}^n \xi_i I_{\{\Delta_i\tilde N^{(\ell)}\ge 1\}} << \theta_2$, for $\ell = 1,2$ and $\sum_{i=1}^n \xi_i I_{\{\Delta_i\tilde N^{(1)}\ge 1,\Delta_i\tilde N^{(2)}\ge 1\}} << \theta_2$. Using the independence of ξ_i on $\Delta_i\tilde N^{(\ell)}$, we reach that

$$E_{i-1}[\xi_i I_{\{\Delta_i\tilde N^{(\ell)}\ge 1\}}] = KE[\xi_i]\theta_\ell, \quad E_{i-1}[\xi_i^2 I_{\{\Delta_i\tilde N^{(\ell)}\ge 1\}}] \le KE[\xi_i^2]\theta_\ell.$$

Thus, if, for any ℓ, we call

$$\sum_{i=1}^{n} \frac{\xi_i I_{\{\Delta_i \tilde{N}^{(\ell)} \geq 1\}}}{\theta_2} \doteq \sum_{i=1}^{n} \chi_i,$$

we have that $\forall t \geq 0$, $\sum_{t_i \leq t} E_{i-1}[\chi_i] \leq K \sum_{t_i \leq t} E[\xi_i] \leq KnE[\xi_1]$, which, by looking at Theorem 4.3, tends to zero in all the cases $\gamma \in [0,1]$. Further, $\sum_{t_i \leq t} E_{i-1}[\chi_i]$ is positive for all t, and increasing in t, thus the convergence is also ucp. Moreover $\forall t \geq 0$, $\sum_{t_i \leq t} E_{i-1}[\chi_i^2] \leq nE[\xi_1^2]/\theta_2 \leq nVar(\xi_1)/(K\theta_2)$, with $K \in (0,1)$, having used that, since ξ_1 is not constant, then $E^2[\xi_i] < E[\xi_i^2]$. Using now for $nVar(\xi_1)$ the expressions at the denominators of (30), (31), (32) it is verified that under our assumptions $nVar(\xi_1)/\theta_2 \to 0$ in all the cases $\gamma \in [0,1]$. We remark that for the case $\gamma \in (0,1)$ and $\alpha_1 \geq x_\star$ condition $u > 1/[2(1+\alpha_2/\alpha_1)]$ is needed, however it is implied by our assumption (7). It follows that $\sum_{i=1}^n \chi_i \stackrel{ucp}{\to} 0$, that is $\sum_{i=1}^n \xi_i I_{\{\Delta_i \tilde{N}^{(\ell)} \geq 1\}} << \theta_2$.

If we now call $P\{\Delta_i \tilde{N}^{(1)} \geq 1, \Delta_i \tilde{N}^{(2)} \geq 1\} \doteq \theta_{1,2} \leq \theta_2$, and

$$\sum_{i=1}^{n} \chi_i \doteq \sum_{i=1}^{n} \frac{\xi_i I_{\{\Delta_i \tilde{N}^{(1)} \geq 1, \Delta_i \tilde{N}^{(2)} \geq 1\}}}{\theta_2},$$

we have $\sum_{t_i \leq t} E_{i-1}[\chi_i] = \left[\frac{t}{h}\right] E[\xi_1] \frac{\theta_{1,2}}{\theta_2} \leq \left[\frac{t}{h}\right] E[\xi_1] \stackrel{ucp}{\to} 0$, and $\sum_{t_i \leq t} E_{i-1}[\chi_i^2] \leq KnVar(\xi_1)/\theta_2 \to 0$, so again $\sum_{i=1}^n \chi_i \stackrel{ucp}{\to} 0$ and $\sum_{i=1}^n \xi_i I_{\{\Delta_i \tilde{N}^{(1)} \geq 1, \Delta_i \tilde{N}^{(2)} \geq 1\}} << \theta_2$.

We now show that within $I_{4,3}$ is negligible either wrt θ_2 or wrt $\sum_{i=1}^n \xi_i$. Each term of the sum is counted only if both $\Delta_i \tilde{N}^{(j)} = 0, j = 1,2$ but $|\Delta_i X^{(\ell)}| > \sqrt{r_h}$ for at least one index ℓ. Note that if $\Delta_i \tilde{N}^{(\ell)} = 0$ then $\Delta_i J^{(\ell)} = 0$ and $\Delta_i \tilde{V}^{(\ell)} = 0$. However, as commented for $I_{2,3}$, we have $\{|\Delta_i X^{(\ell)}| > \sqrt{r_h}, \Delta_i J^{(\ell)} = 0\} \subset \{|\Delta_i M^{(\ell)}| > \sqrt{r_h}(1-p)\}$, and $P\{\Delta_i \tilde{V}^{(\ell)} = 0, |\Delta_i M^{(\ell)}| > \sqrt{r_h}(1-p)\} \leq P\{\Delta_i \tilde{V}^{(\ell)} = 0, |\Delta_i M^{(\ell)}| > \sqrt{r_h}\} + P\{|\Delta_i M^{(\ell)}| \in (\sqrt{r_h}(1-p), \sqrt{r_h}]\} \leq \theta_2^{4/3} + \theta_2 h^\phi \sim \theta_2 h^\phi$. It follows that

$$E[|I_{4,3}|] \leq \sum_{i=1}^{n} ||\xi_i||_2 \sqrt{\theta_2 h^\phi} \leq K\sqrt{n\ Var(\xi_i)}\sqrt{n}\ \theta_2^{\frac{1}{2}} h^{\frac{\phi}{2}} = K\sqrt{n\ Var(\xi_i)}\ \varepsilon^{-\frac{\alpha_2}{2}} h^{\frac{\phi}{2}} \doteq a_n :$$

looking at (30), (31), (32), depending on the different choices of γ, α_1, α_2 we have the following: for $\gamma \in (0,1)$ and $\alpha_1 \leq x_\star$, we have $a_n << \theta_2$ iff $u > 1/(4-\alpha_1)$, however this last condition is implied by (7); for $\gamma \in (0,1)$ and $\alpha_1 > x_\star$ then, using also Proposition 4.4, $a_n << \sum_{i=1}^n \xi_i$; if $\gamma = 1$ then $a_n << \theta_2$ iff $u > 1/(4-\alpha_1)$; if $\gamma = 0$ and either $\alpha_1 < \alpha_2 u$ or $(\alpha_1 = \alpha_2 u, \alpha_2 = 1)$ then $a_n << \theta_2$; if $\gamma = 0$ and either $(\alpha_1 = \alpha_2 u, \alpha_2 > 1)$ or $\alpha_1 > \alpha_2 u$ then $a_n << \sum_{i=1}^n \xi_i$.

Finally we show that $I_{4,4}$ is negligible wrt to θ_2 : we check this when the summands satisfy the three cases $(\Delta_i \tilde{N}^{(2)} = 0, \Delta_i \tilde{N}^{(1)} \geq 1)$; $(\Delta_i \tilde{N}^{(2)} \geq 1, \Delta_i \tilde{N}^{(1)} = 0)$; $(\Delta_i \tilde{N}^{(2)} \geq 1, \Delta_i \tilde{N}^{(1)} \geq 1)$, which are dealt with similarly. For the indices i such that $\Delta_i \tilde{N}^{(2)} = 0$ and $\Delta_i \tilde{N}^{(1)} \geq 1$, then the terms with $\Delta_i J^{(1)} \neq 0$, as previously, do not contribute to $I_{4,4}/\theta_2$, since $|\Delta_i X^{(1)}| \leq \sqrt{r_h}$, and thus $|\Delta_i M^{(1)}| > \sqrt{r_h}(1-p)$. We then remain with the terms where $\Delta_i J^{(1)} = 0$ and, since $|\Delta_i X^{(1)}| \leq \sqrt{r_h}$, we have $|\Delta_i M^{(1)}| \leq \sqrt{r_h}(1+p)$. On the other hand on $\{\Delta_i \tilde N^{(2)}=0\}$ we have $\Delta_i J^{(2)}=0$, and thus also $|\Delta_i M^{(2)}|\le \sqrt{r_h}(1+p)$. It follows that, as for (27), $\Delta_i M^{(\ell)}=\Delta_i M^{(\ell)h}-h\int_{2v_h}^1 x\nu^{(\ell)}(dx)$. Now

$$\frac{1}{\theta_2}E\Big[\Big|\sum_{i=1}^n \Delta_i M^{(1)h}\Delta_i M^{(2)h}I_{\{\Delta_i \tilde N^{(1)}\ge 1\}}\Big|\Big]\le \frac{1}{\theta_2}\Big|\Big|\sum_{i=1}^n \Delta_i M^{(1)h}\Delta_i M^{(2)h}I_{\{\Delta_i \tilde N^{(1)}\ge 1\}}\Big|\Big|_2=$$

$$\frac{1}{\theta_2}\sqrt{\sum_{i=1}^n E[(\Delta_i M^{(1)h}\Delta_i M^{(2)h})^2]P\{\Delta_i\tilde N^{(1)}\ge 1\}}\le \frac{1}{\theta_2}\sqrt{nh^2 r_h^{1-\frac{\alpha_1}{4}-\frac{\alpha_2}{4}}\theta_1}\le\sqrt{r_h^{1-\frac{\alpha_1}{4}+\frac{3}{4}\alpha_2}}\to 0,$$

having used: the independence among the increments and the independence of the $\Delta_i M^{(\ell)h}$ with $\tilde N^{(1)}$, the Hölder inequality to reach that $E[(\Delta_i M^{(1)h})^2(\Delta_i M^{(2)h})^2]\le \int_{t_{i-1}}^{t_i}\int_{x\le v_h}x_1^2\nu^{(1)}(dx_1)\int_{t_{i-1}}^{t_i}\int_{x\le v_h}x_2^2\nu^{(2)}(dx_2)=h^2 r_h^{1-\frac{\alpha_1}{4}-\frac{\alpha_2}{4}}$. Further

$$\frac{1}{\theta_2}E\Big[\Big|\sum_{i=1}^n h^2\int_{2v_h}^1 x\nu^{(2)}(dx)\int_{2v_h}^1 x\nu^{(2)}(dx)I_{\{\Delta_i\tilde N^{(1)}\ge 1\}}\Big|\Big]\le Kh\prod_{\ell=1,2}\Big[|1-(2v_h)^{1-\alpha_\ell}|I_{\alpha_\ell\ne 1}+I_{\alpha_\ell=1}\ln\frac{1}{2v_h}\Big]$$

which in the worst case of $\alpha_1,\alpha_2>1$ is dominated by $Khv_h^{1-\alpha_1}v_h^{1-\alpha_2}\le h^{1+\frac{u}{4}(1-\alpha_1)+\frac{u}{4}(1-\alpha_2)}\to 0$. It follows that $\frac{1}{\theta_2}E\Big[\Big|\sum_{i=1}^n\Delta_i M^{(1)}\Delta_i M^{(2)}I_{\{\Delta_i\tilde N^{(1)}\ge 1\}}\Big|\Big]\to 0$, and thus

$$E\Big[\frac{1}{\theta_2}\Big|\sum_{i=1}^n\Delta_i M^{(1)}\Delta_i M^{(2)}I_{\{\Delta_i\tilde N^{(2)}=0,\Delta_i\tilde N^{(1)}\ge 1,\Delta_i J^{(1)}=0\}\cap\{|\Delta_i X^{(1)}|\le\sqrt{r_h},|\Delta_i X^{(2)}|\le\sqrt{r_h},\}}\Big|\Big]\to 0.$$

For the indices i such that $\Delta_i\tilde N^{(2)}\ge 1$ and $\Delta_i\tilde N^{(1)}=0$, we reason similarly as above and obtain that

$$E\Big[\frac{1}{\theta_2}\sum_{i=1}^n|\Delta_i M^{(1)}\Delta_i M^{(2)}|I_{\{\Delta_i\tilde N^{(2)}\ge 1,\Delta_i J^{(2)}=0,\Delta_i\tilde N^{(1)}=0\}\cap\{|\Delta_i X^{(1)}|\le\sqrt{r_h},|\Delta_i X^{(2)}|\le\sqrt{r_h},\}}\Big]\to 0.$$

For the indices i such that $\Delta_i\tilde N^{(1)}\ge 1,\Delta_i\tilde N^{(2)}\ge 1$, then the terms with one $\Delta_i J^{(\ell)}\ne 0$, are negligible and we remain with the terms where both $\Delta_i J^{(\ell)}=0$, thus we reach that both $|\Delta_i M^{(\ell)}|\le\sqrt{r_h}(1+p)$ and, as above,

$$E\Big[\frac{1}{\theta_2}\sum_{i=1}^n|\Delta_i M^{(1)}\Delta_i M^{(2)}|I_{\{\Delta_i\tilde N^{(2)}\ge 1,\Delta_i\tilde N^{(1)}\ge 1\}\cap\{|\Delta_i X^{(1)}|\le\sqrt{r_h},|\Delta_i X^{(2)}|\le\sqrt{r_h},\}}\Big]\to 0,$$

and the proof of the negligibility of $I_{4,4}$ wrt θ_2 is completed.

We thus obtained that $\hat{IC}-IC\sim\sqrt{h}+\sum_{i=1}^n\xi_i+\theta_2$. Now we are going to make this more explicit. In (5) we compared $\sqrt{h}$ with $\sum_{i=1}^n\xi_i$. As for θ_2 versus $\sqrt{h}$ we have that:

$$\theta_2<<\sqrt{h}\text{ if }\alpha_2<\tfrac{1}{2u};\ \theta_2\sim\sqrt{h}\text{ if }\alpha_2=\tfrac{1}{2u};\ \theta_2>>\sqrt{h}\text{ if }\alpha_2>\tfrac{1}{2u}.$$

Comparing now θ_2 with $\sum_{i=1}^n\xi_i$, we reach that

when $\gamma=1$	$\theta_2>>\sum_{i=1}^n\xi_i$	for $\alpha_2=\alpha_1=1$: if $u>\frac14$; for $(\alpha_1,\alpha_2)\ne(1,1)$: $\forall u\in(0,\frac12)$
when $\gamma\in[0,1)$	$\theta_2>>\sum_{i=1}^n\xi_i$	for $\alpha_1\le\alpha_2 u$: any $u\in(0,\frac12)$; for $\alpha_2>\alpha_1>\alpha_2 u$: iff $u>\frac{1}{1+\frac{\alpha_2}{\alpha_1}}$
when $\gamma\in[0,1)$	$\theta_2<<\sum_{i=1}^n\xi_i$	for $\alpha_1=\alpha_2$: any $u\in(0,\frac12)$.

It follows that

$$\hat{IC} - IC \sim I_{\alpha_2 \geq \frac{1}{2u}} \left(\theta_2 \Big[I_{\{\gamma=1\}} + I_{\{\gamma\in[0,1),\alpha_1\leq\alpha_2 u\}} + I_{\{\gamma\in[0,1),\alpha_2>\alpha_1>\alpha_2 u, u\geq\frac{\alpha_1}{\alpha_1+\alpha_2}\}} \Big] \right.$$

$$\left. + \sum_{i=1}^n \xi_i \Big[I_{\{\gamma\in[0,1),\alpha_2>\alpha_1>\alpha_2 u, u<\frac{\alpha_1}{\alpha_1+\alpha_2}\}} + I_{\{\gamma\in[0,1),\alpha_2=\alpha_1>\alpha_2 u\}} \Big] \right)$$

$$+ I_{\alpha_2<\frac{1}{2u}} \left(\sqrt{h} \Big[I_{\{\gamma=1,\alpha_1<\alpha_1^{\star\star}\}} + I_{\{\gamma\in[0,1),\alpha_1\leq\alpha_1^\star\}} \Big] + \sum_{i=1}^n \xi_i \Big[I_{\gamma=1,\alpha_1\geq\alpha_1^{\star\star}} + I_{\{\gamma\in[0,1),\alpha_1>\alpha_1^\star\}} \Big] \right).$$

However: note that $u < \frac{\alpha_1}{\alpha_1+\alpha_2}$ implies $\alpha_1 > \alpha_2 u$; if $\alpha_1 = \alpha_2$ then $\alpha_1 > \alpha_2 u$, since $\alpha_2 > \alpha_2 u$; $\alpha_1 < 1/(2u) \Rightarrow \alpha_1 \leq \alpha_1^{\star\star}$. Thus the above display simplifies and (10, 11, 12) follow. □

5 Appendix 2

This appendix contains the technical proofs of the following results presented in Section 3 and in Appendix 1: statement (5), Remark x) to Theorem 3.2, Theorem 4.3, Proposition 4.4, Theorem 4.6, Lemma 4.8, Lemma 4.9, Lemma 4.10 and Lemma 4.11.

5.1 Remarks for the main result

Statement (5). Defined

$$\alpha_1^\star \doteq \frac{\alpha_2 u}{\alpha_2 u - u + 1/2} \in (2u, 1), \quad \alpha_1^{\star\star} \doteq \frac{1 + 2u(2-\alpha_2)}{2u} > \frac{1}{2u} > 1,$$

we have that:

$$5 \begin{cases} \text{if } \gamma \in [0,1): & \sum_i \xi_i << \sqrt{h} \quad \text{iff } \alpha_1 < \alpha_1^\star; \\ \text{if } \gamma = 1: & \sum_i \xi_i << \sqrt{h} \quad \text{iff } \alpha_1 < \alpha_1^{\star\star}. \end{cases} \tag{29}$$

Proof . We heavily use Proposition 4.4. In the case $\gamma \in [0,1)$ we have $\sum_{i=1}^n \xi_i \sim nE[\xi_1]$. Using (34) we have that on $\{\alpha_1 \leq \alpha_2 u, \alpha_2 = 1\} \cup \{\alpha_1 \leq \alpha_2 u, \alpha_2 > 1\}$ both $\alpha_1 < \alpha_1^\star$ and $nE[\xi_1]/\sqrt{h} \to 0$. On $\{\alpha_1 > \alpha_2 u\}$ then $nE[\xi_1]/\sqrt{h} \to 0$ iff $\alpha_1 < \alpha_1^\star$.
In the case $\gamma = 1$ then on $\{\alpha_1 < 1, \alpha_2 \geq 1\} \cup \{\alpha_1 = 1 < \alpha_2\}$ we have $\alpha_1 < \alpha_1^{\star\star}$. If $u > 1/(2 + \alpha_2 - \alpha_1)$ then $\sum_{i=1}^n \xi \sim nE[\xi_1]$ and $nE[\xi_1]/\sqrt{h} \to 0$; if $u \leq 1/(2 + \alpha_2 - \alpha_1)$ then $\sum_{i=1}^n \xi_i/\sqrt{h} \sim \sqrt{nVar(\xi_1)}/\sqrt{h} \to 0$. On $\{1 < \alpha_1 \leq \alpha_2\}$: if $u > \frac{1}{\alpha_1+\alpha_2}$ then $\sum_{i=1}^n \xi \sim nE[\xi_1] << \sqrt{h}$ iff $\alpha_1 < \alpha_1^{\star\star}$. On the other hand $u \leq \frac{1}{\alpha_1+\alpha_2}$ is equivalent to $\alpha_1 \leq 1/u - \alpha_2$, which is less than $\alpha_1^{\star\star}$, and if $u \leq \frac{1}{\alpha_1+\alpha_2}$ then $\sum_{i=1}^n \xi \sim \sqrt{nVar(\xi_1)} << \sqrt{h}$. Finally, when $\alpha_1 = \alpha_2 = 1$ then $\sum_{i=1}^n \xi_i/\sqrt{h} \sim \sqrt{nVar(\xi_1)}/\sqrt{h} = \sqrt{h\varepsilon^2}/\sqrt{h} \to 0$, and $\alpha_1 = 1 < \alpha^{\star\star}$. □

Remark x) to Theorem 3.2. For fixed h, the convergence speed is a function $s(\gamma, \alpha_1, \alpha_2, u)$ of our parameters. Such a function is smooth most of the times, however it has some singularities (as is evident in Figure 1).

In fact when $u \geq \alpha_1/(\alpha_1+\alpha_2)$ and $\gamma \in [0,1)$: if $\alpha_1 \neq \alpha_2$ but the two indices are close and above $1/(2u)$, then $s = h\varepsilon^{-\alpha_2} = h^{1-\alpha_2 u}$ while at $\alpha_1 = \alpha_2$ the function s jumps at $\varepsilon^{2-\alpha_2} = h^{2u-\alpha_2 u}$. The jump would disappear if it was $u = 1/2$.
On the contrary, we have smoothness at $\alpha_1 = \alpha_1^\star$ if $\alpha_2 < 1/(2u)$: in fact if α_1 is much less than α_2 (case $\alpha_1 \leq \alpha_1^\star < 1 \leq \alpha_2 < \frac{1}{2u}$) then $s = h^{1/2}$; for α_1 at $\alpha_1^\star$ we have $s = \sqrt{h} = \varepsilon^{1+\frac{\alpha_2}{\alpha_1}-\alpha_2}$, and with $\alpha_1 \in (\alpha_1^\star, \alpha_2]$ still is $s = \varepsilon^{1+\frac{\alpha_2}{\alpha_1}-\alpha_2}$. When $\gamma = 1$ we have smoothness at $\alpha_2 = 1/(2u)$: in fact when $\alpha_2 = 1/(2u)$ we have $\sqrt{h} = h\varepsilon^{-\alpha_2}$. □

5.2 Proofs of the tools for Theorem 3.1

Theorem 4.3. Assume **A2-A5**, $0 < \alpha_1 \leq \alpha_2 < 2, \alpha_2 \geq 1, 0 < c_1 \leq c_2$. Take $\varepsilon = h^u$, any $u \in]0, \frac{1}{2}[$ and define

$$x_\star \doteq \frac{1 + 2u - \sqrt{-4(2\alpha_2 - 1)u^2 + 4u + 1}}{2u} \in (\alpha_2 u, \alpha_2).$$

Then as $\varepsilon \to 0$ the following quotients are tight:
i) if $\gamma \in (0,1)$:

$$\frac{\sum_i \xi_i - T(1-\gamma)C(1,1)\varepsilon^{1+\frac{\alpha_2}{\alpha_1}-\alpha_2} I_{\{\alpha_1 > \alpha_2 u\} \cup \{\alpha_1 = \alpha_2 u, \alpha_2 > 1\}} - Thc_{A_1}c_{A_2}F_0(\varepsilon)}{\sqrt{T}\varepsilon^{1-\alpha_2/2}\sqrt{h\varepsilon^{2-\alpha_1}\gamma C_1(2)C(0,2)I_{\{\alpha_1 \leq x_\star\}} + \varepsilon^{2\frac{\alpha_2}{\alpha_1}}(1-\gamma)C(2,2)I_{\{\alpha_1 \geq x_\star\}}}} \tag{30}$$

ii) If $\gamma = 1$:

$$\frac{\sum_i \xi_i - Thc_{A_1}c_{A_2}F_1(\varepsilon)}{\sqrt{T}\sqrt{h}\varepsilon^{2-\alpha_1/2-\alpha_2/2}\sqrt{C_1(2)C_2(2)}}, \tag{31}$$

iii) If $\gamma = 0$: with $G \doteq C(2,2) - 2c_{A_1}C(1,2) + c_{A_1}^2 C(0,2)$ we have

$$\frac{\sum_i \xi_i - TC(1,1)\varepsilon^{1+\frac{\alpha_2}{\alpha_1}-\alpha_2} I_{\{\alpha_1 > \alpha_2 u\} \cup \{\alpha_1 = \alpha_2 u, \alpha_2 > 1\}} - Thc_{A_1}c_{A_2}F_0(\varepsilon)}{\sqrt{T}\varepsilon^{1-\alpha_2/2}\sqrt{h^2 c_{A_1}^2 C(0,2)I_{\{\alpha_1 < \alpha_2 u\}} + \varepsilon^{2\frac{\alpha_2}{\alpha_1}}\Big[C(2,2)I_{\{\alpha_1 > \alpha_2 u\}} + GI_{\{\alpha_1 = \alpha_2 u\}}\Big]}}. \tag{32}$$

Proof . Define

$$X_m^\varepsilon \doteq \int_0^h \int_{|x| \leq \varepsilon} x \tilde{\mu}^{(m)}(dx, dt)$$

and recall A_m^ε in iv) of Lemma 4.2: each $\xi_i,\ i = 1..n$, has the same law as $(X_1^\varepsilon - hA_1^\varepsilon)(X_2^\varepsilon - hA_2^\varepsilon)$. For simplicity we write A_m in place of A_m^ε. We are going to compute $E[\sum_{i=1}^n \xi_i]$ and $Var[\sum_{i=1}^n \xi_i]$, we thus need to compute the moments $E[(X_1^\varepsilon)^k (X_2^\varepsilon)^m]$, with $k = 2,1,0, m = 2,1,0$. The bivariate process $X^\varepsilon = (X_1^\varepsilon, X_2^\varepsilon)$ is Lévy with Lévy measure $\nu_\varepsilon(dx_1, dx_2) = I_{\{0 \leq x_1, x_2 \leq \varepsilon\}}\nu_\gamma(dx_1, dx_2)$, and note that, for small ε, $0 \leq x_1, x_2 \leq \varepsilon \Rightarrow x_1^2 + x_2^2 \leq 1$, so we reach the desired moments by differentiating the characteristic function $\varphi(u_1, u_2) = E[e^{iu_1 X_1^\varepsilon + iu_2 X_2^\varepsilon}] = \exp\{h \int (e^{iu_1 x_1 + iu_2 x_2} - 1 - iu_1 x_1 - iu_2 x_2)\ \nu_\varepsilon(dx_1, dx_2)\}$, then evaluating it at $(0,0)$, recalling the expression of ν_γ and using Lemma 4.2. In particular we have:

$$E\Big[X_1^\varepsilon\Big] = E\Big[X_2^\varepsilon\Big] = 0$$

$$E\Big[\Big(X_1^\varepsilon\Big)^2\Big] = h\int_{\mathbb{R}^2} x_1^2\nu_\varepsilon(dx_1,dx_2) = \gamma C_1(2)h\varepsilon^{2-\alpha_1} + (1-\gamma)C(2,0)h\varepsilon^{2\frac{\alpha_2}{\alpha_1}-\alpha_2}.$$

Note that if $\gamma \in (0,1)$ then as $\varepsilon \to 0$ we have

$$E\Big[\Big(X_1^\varepsilon\Big)^2\Big] = \gamma C_1(2)h\varepsilon^{2-\alpha_1} + (1-\gamma)C(2,0)h\varepsilon^{2\frac{\alpha_2}{\alpha_1}-\alpha_2} \sim h\varepsilon^{2-\alpha_1}\mathcal{A},$$

where $\mathcal{A} = \gamma C_1(2)I_{\{\alpha_1\le\alpha_2\}} + (1-\gamma)C(2,0)I_{\{\alpha_1=\alpha_2\}}$. In fact, with $\phi \doteq \frac{\alpha_2}{\alpha_1} \in [1,+\infty)$, the quotient $\varepsilon^{2\frac{\alpha_2}{\alpha_1}-\alpha_2}/\varepsilon^{2-\alpha_1}\varepsilon^{2\phi-\alpha_1\phi-2+\alpha_1} = \varepsilon^{(2-\alpha_1)(\phi-1)}$ has an exponent which is non-negative for all $\alpha_1,\alpha_2 \in (0,2)$, and zero for $\alpha_1 = \alpha_2$.

$$E\Big[\Big(X_2^\varepsilon\Big)^2\Big] = h\int_{\mathbb{R}^2} x_2^2\nu_\varepsilon(dx_1,dx_2) = hC(0,2)\varepsilon^{2-\alpha_2}$$

$$E\Big[X_1^\varepsilon X_2^\varepsilon\Big] = h\int_{\mathbb{R}^2} x_1x_2\nu_\varepsilon(dx_1,dx_2) = h\varepsilon^{1+\frac{\alpha_2}{\alpha_1}-\alpha_2}C(1,1)(1-\gamma)$$

$$E\Big[(X_1^\varepsilon)^2X_2^\varepsilon\Big] = h\int_{\mathbb{R}^2} x_1^2x_2\nu_\varepsilon(dx_1,dx_2) = h\varepsilon^{1+2\frac{\alpha_2}{\alpha_1}-\alpha_2}C(2,1)(1-\gamma)$$

$$E\Big[X_1^\varepsilon(X_2^\varepsilon)^2\Big] = h\int_{\mathbb{R}^2} x_1x_2^2\nu_\varepsilon(dx_1,dx_2) = h\varepsilon^{2+\frac{\alpha_2}{\alpha_1}-\alpha_2}C(1,2)(1-\gamma)$$

$$E\Big[(X_1^\varepsilon)^2(X_2^\varepsilon)^2\Big] = 2E^2\Big[X_1^\varepsilon X_2^\varepsilon\Big] + h\int_{\mathbb{R}^2} x_1^2x_2^2\nu_\varepsilon(dx_1,dx_2) + h^2\int_{\mathbb{R}^2} x_1^2\nu_\varepsilon(dx_1,dx_2)\cdot$$
$$\cdot\int_{\mathbb{R}^2} x_2^2\nu_\varepsilon(dx_1,dx_2) \sim (1-\gamma)h\varepsilon^{2+2\frac{\alpha_2}{\alpha_1}-\alpha_2}C(2,2) + hC(0,2)\varepsilon^{2-\alpha_2}E\Big[\Big(X_1^\varepsilon\Big)^2\Big].$$

Let us first concentrate on $E[\sum_i \xi_i]$. From the above we reach that

$$E[\xi_i]=E[X_1^\varepsilon X_2^\varepsilon]+h^2A_1A_2=(1-\gamma)C(1,1)h\varepsilon^{1+\frac{\alpha_2}{\alpha_1}-\alpha_2}+c_{A_1}c_{A_2}h^2\Big[(1-\varepsilon^{1-\alpha_1})(1-\varepsilon^{1-\alpha_2})I_{\alpha_1,\alpha_2\neq 1}$$
$$+\ln\frac{1}{\varepsilon}(1-\varepsilon^{1-\alpha_2})I_{\alpha_1=1<\alpha_2} + (1-\varepsilon^{1-\alpha_1})\log\frac{1}{\varepsilon}I_{\{\alpha_1<\alpha_2=1\}} + \ln^2\frac{1}{\varepsilon}I_{\alpha_1=\alpha_2=1}\Big]. \tag{33}$$

Note that since $\varepsilon = h^u$, as $h \to 0$ we have $E[\xi_i] \to 0$.

i) and iii). If $\gamma \in [0,1)$, then we have the following leading terms in the expression of $E[\xi_i]$, when $h \to 0$: when both $\alpha_m = 1$, for sufficiently small h we have $h\varepsilon^{1+\frac{\alpha_2}{\alpha_1}-\alpha_2} >> h^2\ln^2\frac{1}{\varepsilon}$ so the leading term is $h\varepsilon^{1+\frac{\alpha_2}{\alpha_1}-\alpha_2}$, coming from $E[X_1^\varepsilon X_2^\varepsilon]$; when $\alpha_1 = 1 < \alpha_2$, then the leading term is still $h\varepsilon^{1+\frac{\alpha_2}{\alpha_1}-\alpha_2}$; when $\alpha_1 < \alpha_2 = 1$, the leading term is $h\varepsilon^{1+\frac{\alpha_2}{\alpha_1}-\alpha_2}$ when $u = \alpha_2 u < \alpha_1$, while is $h^2(1-\varepsilon^{1-\alpha_1})\log\frac{1}{\varepsilon} \sim h^2\log\frac{1}{\varepsilon}$, coming from $h^2A_1A_2$, otherwise. When both $\alpha_m \neq 1$ then under our framework we necessarily have $\alpha_2 > 1$; note that $\alpha_2 u < 1$; if $\alpha_1 > 1$ then the leading term turns out to be $h\varepsilon^{1+\frac{\alpha_2}{\alpha_1}-\alpha_2}$; If $\alpha_1 < 1$: $h\varepsilon^{1+\frac{\alpha_2}{\alpha_1}-\alpha_2}$ is the only leading term only if $\alpha_2 u < \alpha_1$; when $\alpha_2 u = \alpha_1$ (and still $\alpha_2 > 1$) then $h\varepsilon^{1+\frac{\alpha_2}{\alpha_1}-\alpha_2} \sim h^2(1-\varepsilon^{1-\alpha_1})(1-\varepsilon^{1-\alpha_2}) \sim -h^2\varepsilon^{1-\alpha_2}$; when $\alpha_2 u > \alpha_1$ then the leading term is $-h^2\varepsilon^{1-\alpha_2}$. However $\{\alpha_1 = \alpha_2 = 1\} \cup \{\alpha_1 = 1 < \alpha_2\} \cup \{u < \alpha_1 < \alpha_2 = 1\} \cup \{\alpha_1 \neq 1, \alpha_2 > 1, \alpha_2 u < \alpha_1\} = \{\alpha_1 > \alpha_2 u\}$ and here is where the only leading term is $E\Big[X_1^\varepsilon X_2^\varepsilon\Big]$;

$\{\alpha_1 = u, \alpha_2 = 1\} \cup \{\alpha_1 = \alpha_2 u < 1 < \alpha_2\} = \{\alpha_1 = \alpha_2 u\}$ and here: if $\alpha_2 > 1$ then $E\Big[X_1^\varepsilon X_2^\varepsilon\Big]$ and $h^2 A_1 A_2$ have the same speed $h\varepsilon^{1+\frac{\alpha_2}{\alpha_1}-\alpha_2}$; if $\alpha_2 = 1$ then only $h^2 A_1 A_2 \sim h^2 \log\frac{1}{\varepsilon}$ is leading;
$\{\alpha_1 < \alpha_2 u < 1 = \alpha_2\} \cup \{\alpha_1 < \alpha_2 u < 1 < \alpha_2\} = \{\alpha_1 < \alpha_2 u\}$ and here the only leading term is $h^2 A_1 A_2$. Thus

$$E[\sum_i \xi_i] \sim T(1-\gamma)C(1,1)\varepsilon^{1+\frac{\alpha_2}{\alpha_1}-\alpha_2} I_{\{\alpha_1>\alpha_2 u\}\cup\{\alpha_1=\alpha_2 u, \alpha_2>1\}} - Thc_{A_1}c_{A_2}F_0(\varepsilon). \tag{34}$$

ii) If $\gamma = 1$, then $nE[\xi_1] = nh^2 A_1 A_2$, and again the leading term is different for different choices of α_1, α_2. We have

$$nE[\xi_1] \sim Thc_{A_1}c_{A_2}F_1(\varepsilon). \tag{35}$$

As for $Var(\xi_i)$: in the general case $\gamma \in [0,1]$, writing X_m for X_m^ε, $Var(\xi_i)$ is given by

$$\begin{aligned} &E[X_1^2X_2^2] - 2hA_2E[X_1^2X_2] - 2hA_1E[X_1X_2^2] + h^2A_2^2E[X_1^2] + h^2A_1^2E[X_2^2] + 2h^2A_1A_2E[X_1X_2] + \\ &-E^2[X_1X_2] = h^2\int_{0\le x_1,x_2\le\varepsilon} x_1^2 d\nu \int_{0\le x_1,x_2\le\varepsilon} x_2^2 d\nu + E^2[X_1X_2] + h\int_{0\le x_1,x_2\le\varepsilon} x_1^2x_2^2 d\nu + \\ &-2hA_2E[X_1^2X_2] - 2hA_1E[X_1X_2^2] + h^2A_2^2E[X_1^2] + h^2A_1^2E[X_2^2] + 2h^2A_1A_2E[X_1X_2] \\ &\doteq \sum_{\ell=1}^8 V_\ell, \end{aligned} \tag{36}$$

where

$$V_1 \doteq h^2\int_{0\le x_1,x_2\le\varepsilon} x_1^2 d\nu \int_{0\le x_1,x_2\le\varepsilon} x_2^2 d\nu;\ V_2 \doteq E^2[X_1X_2];\ V_3 \doteq h\int_{0\le x_1,x_2\le\varepsilon} x_1^2x_2^2 d\nu;$$
$$V_4 \doteq -2hA_2E[X_1^2X_2]; V_5 \doteq -2hA_1E[X_1X_2^2];\ V_6 \doteq h^2A_2^2E[X_1^2];$$
$$V_7 \doteq h^2A_1^2E[X_2^2];\ V_8 \doteq 2h^2A_1A_2E[X_1X_2].$$

As $\varepsilon \to 0$ all these terms tend to zero: we now establish the leading ones and we only keep them.

i) If $\gamma \in (0,1)$, we have the following properties:

$$V_1 \sim h^2\varepsilon^{4-\alpha_1-\alpha_2}\mathcal{A}C(0,2) >> V_6 \sim h^3c_{A_2}^2[(1-\varepsilon^{1-\alpha_2})^2 I_{\alpha_2\neq 1} + \ln^2\frac{1}{\varepsilon}\ I_{\alpha_2=1}]\mathcal{A}\varepsilon^{2-\alpha_1};\ V_2 = (1-\gamma)^2\cdot$$
$$\cdot C^2(1,1)h^2\varepsilon^{2(\frac{\alpha_2}{\alpha_1}+1-\alpha_2)},\ V_4 = -2(1-\gamma)h^2c_{A_2}[(1-\varepsilon^{1-\alpha_2})I_{\alpha_2\neq 1} + \ln\frac{1}{\varepsilon}\ I_{\alpha_2=1}]C(2,1)\varepsilon^{1+2\frac{\alpha_2}{\alpha_1}-\alpha_2}$$

are negligible wrt $V_3 = (1-\gamma)C(2,2)h\varepsilon^{2+2\frac{\alpha_2}{\alpha_1}-\alpha_2}$; recalling that we chose $\alpha_1 \le \alpha_2$ and we only are interested in the case where at least $\alpha_2 \ge 1$, we have that

$$V_8 = 2(1-\gamma)C(1,1)c_{A_1}c_{A_2}h^3\varepsilon^{1+\frac{\alpha_2}{\alpha_1}-\alpha_2}\Big[(1-\varepsilon^{1-\alpha_1})\cdot$$
$$\cdot(1-\varepsilon^{1-\alpha_2})I_{\alpha_1,\alpha_2\neq 1} + (1-\varepsilon^{1-\alpha_2})\ln\frac{1}{\varepsilon}I_{\alpha_1=1<\alpha_2} + (1-\varepsilon^{1-\alpha_1})\ln\frac{1}{\varepsilon}I_{\alpha_1<1=\alpha_2} + \ln^2\frac{1}{\varepsilon}\ I_{\alpha_1=\alpha_2=1}\Big]$$

$$<< V_5 = -2h^2 c_{A_1}[(1-\varepsilon^{1-\alpha_1})I_{\alpha_1\neq 1} + \ln\frac{1}{\varepsilon}\ I_{\alpha_1=1}](1-\gamma)C(1,2)\varepsilon^{2+\frac{\alpha_2}{\alpha_1}-\alpha_2}.$$

Note that since the terms V_2 and V_8 are both negligible, here we do not need to distinguish which is the leading term within $V_2 + V_8 = E[X_1X_2]\Big(E[X_1X_2] + 2h^2A_1A_2\Big)$. Finally

$$V_7 = h^3 c_{A_1}^2[(1-\varepsilon^{1-\alpha_1})^2 I_{\alpha_1\neq 1} + \ln^2\frac{1}{\varepsilon}\ I_{\alpha_1=1}]C(0,2)\varepsilon^{2-\alpha_2} << V_1,$$

so we are left with

$$Var(\xi_j) \sim V_1 + V_3 + V_5.$$

Now, as $h \to 0$, we have:

$$\frac{V_1}{V_5} \to \begin{cases} 0 & \text{if } \alpha_2 = \alpha_1 = 1 \\ K & \text{if } \alpha_2 = \alpha_1 > 1 \\ \infty & \text{if } \alpha_1 < 1 \leq \alpha_2 \text{ or } \alpha_1 = 1 < \alpha_2 \text{ or } 1 < \alpha_1 < \alpha_2 \end{cases}$$

$$\tfrac{V_3}{V_5} \to \begin{cases} 0 & \text{if } \alpha_1 < \alpha_2 u \\ K & \text{if } \alpha_1 = \alpha_2 u \\ \infty & \text{if } \alpha_1 > \alpha_2 u \end{cases} \qquad \tfrac{V_1}{V_3} \to \begin{cases} 0 & \text{if } \alpha_1 \in (x_\star, 2) \\ K & \text{if } \alpha_1 = x_\star \\ \infty & \text{if } \alpha_1 \in (0, x_\star). \end{cases}$$

By considering the different regions $\alpha_1 < \alpha_2 u$; $\alpha_1 = \alpha_2 u$; $\alpha_1 \in (\alpha_2 u, x_\star)$; $\alpha_1 = x_\star$; $\alpha_1 \in (x_\star, 2)$, we find that V_5 is never the leading term in $V_1 + V_3 + V_5$, V_1 is the only leading term for $\alpha_1 \in (0, x_\star)$; $V_1 \sim V_3$ are leading for $\alpha_1 = x_\star$; and V_3 is the only leading term for $\alpha_1 \in (x_\star, 2)$. However if $\alpha_1 \leq x_\star$ then necessarily $\alpha_1 < \alpha_2$ so $\mathcal{A}$ becomes $\gamma C_1(2)$ and

$$\begin{aligned} Var(\xi_i) &\sim h^2\gamma C_1(2)C(0,2)\varepsilon^{4-\alpha_1-\alpha_2} I_{\{\alpha_1\leq x_\star\}} + h(1-\gamma)C(2,2)\varepsilon^{2+2\frac{\alpha_2}{\alpha_1}-\alpha_2} I_{\{\alpha_1\geq x_\star\}} \\ &= h\varepsilon^{2-\alpha_2}\left[h\varepsilon^{2-\alpha_1}\gamma C_1(2)C(0,2)I_{\{\alpha_1\leq x_\star\}} + \varepsilon^{2\frac{\alpha_2}{\alpha_1}}(1-\gamma)C(2,2)I_{\{\alpha_1\geq x_\star\}}\right] \end{aligned}$$

so, recalling (34), (30) follows.

ii) If $\gamma = 1$, then it turns out that $Var(\xi_1)$ is given by $V_1 + V_6 + V_7 \approx$

$$\approx V_1 = E[X_1^2X_2^2] = h^2\int_{0<x_1\leq\varepsilon} x_1^2\nu_\perp(dx_1)\int_{0<x_1\leq\varepsilon} x_1^2\nu_\perp(dx_1) = h^2\varepsilon^{4-\alpha_1-\alpha_2}C_1(2)C_2(2), \quad (37)$$

and thus, recalling (35), (31) is verified.

iii) If $\gamma = 0$ then $Var(\xi_1) \sim V_3 + V_5 + V_7$ and it turns out that

$$Var(\xi_1) \sim \begin{cases} V_3 & \text{if } \alpha_1 > \alpha_2 u \\ V_3 \sim V_5 \sim V_7 & \text{if } \alpha_1 = \alpha_2 u \\ V_7 & \text{if } \alpha_1 < \alpha_2 u, \end{cases}$$

and, recalling (34), (32) follows. □

Proposition 4.4 Assume $0 < \alpha_1 \leq \alpha_2 < 2$, $\alpha_2 \geq 1$, $0 < c_1 \leq c_2$, $u \in (0, \frac{1}{2})$. As $h \to 0$ we have $\frac{\sqrt{nVar(\xi_1)}}{nE[\xi_1]} \to 0$ in the following cases:

i) for $\gamma \in [0,1)$: for any choices of α_1, α_2 and u, as in the assumptions;
ii) for $\gamma = 1$: on $\{\alpha_1 < 1, \alpha_2 \geq 1\} \cup \{\alpha_1 = 1 < \alpha_2\}$ iff $u \in (\frac{1}{2+\alpha_2-\alpha_1}, \frac{1}{2})$; on $\{1 < \alpha_1 \leq \alpha_2\}$ iff $u \in (\frac{1}{\alpha_1+\alpha_2}, \frac{1}{2})$.
We have $\frac{\sqrt{nVar(\xi_1)}}{nE[\xi_1]} \to +\infty$ in the following case:
iii) for $\gamma = 1$: on $\{\alpha_1 = \alpha_2 = 1\}$, any $u \in (0, \frac{1}{2})$.

Proof i) Case $\gamma \in (0,1)$. We compute $\frac{\sqrt{nVar(\xi_1)}}{nE[\xi_1]}$ by using the information (rate of $nE[\xi_1]$ and of $\sqrt{nVar(\xi_1)}$) summarized in (30) in the four different cases 1) $\alpha_1 \in (0, \alpha_2 u], \alpha_2 > 1$; 2) $\alpha_1 \in (0, \alpha_2 u], \alpha_2 = 1$; 3) $\alpha_1 \in (\alpha_2 u, x_\star]$; 4) $\alpha_1 \in (x_\star, \alpha_2]$. In the cases 1), 2), 3) we have $\alpha_1 \leq x_\star < \alpha_2$, thus $\alpha_1 \neq \alpha_2$, and we reach that a sufficient condition for $\frac{\sqrt{nVar(\xi_1)}}{nE[\xi_1]} \to 0$ is $u \in (\frac{1}{2+\alpha_2-\alpha_1}, \frac{1}{2})$. However $x_\star < 2+\alpha_2-1/u$, thus if $\alpha_1 \leq x_\star$, then $\alpha_1 < 2+\alpha_2-1/u$, which is equivalent to $u > \frac{1}{2+\alpha_2-\alpha_1}$. On the other hand, in the case 4) we reach $\frac{\sqrt{nVar(\xi_1)}}{nE[\xi_1]} \to 0$ for any $u \in (0, 1/2)$.
Case $\gamma = 0$. We now look at (32). Here we separately study the regions $\{\alpha_1 > \alpha_2 u\}$; $\{\alpha_1 = \alpha_2 u\}$; $\{\alpha_1 < \alpha_2 u, \alpha_2 > 1\}$; $\{\alpha_1 < \alpha_2 u, \alpha_2 = 1\}$ and conclude.
ii) and iii). For $\gamma = 1$ we look at (31) and we separately study the regions $\{\alpha_1 < 1 < \alpha_2\}$; $\{\alpha_1 < 1 = \alpha_2\}$; $\{\alpha_1 = 1 < \alpha_2\}$; $\{\alpha_1 = \alpha_2 = 1\}$; and $\{1 < \alpha_1 \leq \alpha_2\}$ and reach the results. $\square$

Theorem 4.6 When $\gamma = 1 = \alpha_1 = \alpha_2$: $\forall\ u \in (0, \frac{1}{2})$, with $\overset{d}{\to}$ denoting convergence in distribution, we have

$$\frac{\sum_{i=1}^n \xi_i - nE[\xi_1]}{\sqrt{nVar(\xi_1)}} \overset{d}{\to} \mathcal{N}.$$

Proof Under $\gamma = 1 = \alpha_1 = \alpha_2$, $M^{(1)}$ and $M^{(2)}$ are independent, and $nVar(\xi_1) \sim h\varepsilon^2$. By the Lindeberg-Feller Theorem, it is sufficient to show that for all $\delta > 0$ we have $nE[\tilde{\xi}_1^2 I_{\{|\tilde{\xi}_1| > \delta\}}] \to 0$. We begin evaluating $P\{|\tilde{\xi}_1| > \delta\}$: by using that we have $\frac{nE[\xi_1]}{\sqrt{nVar(\xi_1)}} \to 0$, $hA_1 = hA_2$ and $X_1^\varepsilon = X_1$ has the same law as $X_2^\varepsilon = X_2$, we obtain

$$P\{|\tilde{\xi}_1| > \delta\} \leq P\Big\{|\xi_1| > \frac{\delta}{2}\sqrt{nVar(\xi_1)}\Big\} = P\Big\{|M_h'^{(1)}||M_h'^{(2)}| > \frac{\delta}{2}\sqrt{nVar(\xi_1)}\Big\}$$

$$\leq P\Big\{|X_1||X_2| + hA_2|X_1| + hA_1|X_2| + h^2 A_1 A_2 > \frac{\delta}{2}\sqrt{nVar(\xi_1)}\Big\} \leq$$

$$P\Big\{|X_1||X_2| > \frac{\delta}{8}\sqrt{nVar(\xi_1)}\Big\} + 2P\Big\{hA_2|X_1| > \frac{\delta}{8}\sqrt{nVar(\xi_1)}\Big\} + P\Big\{h^2 A_1 A_2 > \frac{\delta}{8}\sqrt{nVar(\xi_1)}\Big\}. \tag{38}$$

Now, for sufficiently small h the last term is 0, because $\frac{h^2 A_1 A_2}{\sqrt{nVar(\xi_1)}} = \frac{h^2 \log^2 \frac{1}{\varepsilon}}{\sqrt{h\varepsilon}} = h^{\frac{3}{2}-u} \log^2 \frac{1}{\varepsilon} \to 0$. We now evaluate the other 2 probabilities in (38) to establish their magnitude orders: since $X_1 X_2$ is centered, by the Čebyšëv inequality, used that $\alpha_1 = 1$, we have

$$P\Big\{|X_1 X_2| > \frac{\delta}{8}\sqrt{nVar(\xi_1)}\Big\} \leq \frac{Var\big[|X_1 X_2|\big]}{Kh\varepsilon^2} = \frac{E^2 X_1^2}{Kh\varepsilon^2} = \frac{\big(h\varepsilon^{2-\alpha_1}\big)^2}{Kh\varepsilon^2} = Kh;$$

$$P\Big\{hA_2|X_1| > \frac{\delta}{8}\sqrt{nVar(\xi_1)}\Big\} = P\Big\{|X_1| > \frac{\delta}{8}\frac{h^{u-\frac{1}{2}}}{\log\frac{1}{\varepsilon}}\Big\} \le K\frac{Var(|X_1|)}{h^{2u-1}}\log^2\frac{1}{\varepsilon} \le Kh^{2-u}\log^2\frac{1}{\varepsilon}.$$

Noting that $\frac{h^{2-u}\log^2\frac{1}{\varepsilon}}{h} \to 0$, it follows that $P\{|\tilde{\xi}_1| > \delta\} \le Kh$. Now, for any conjugate exponents p, q,

$$nE[\tilde{\xi}_1^2 I_{\{|\tilde{\xi}_1|>\delta\}}] \le nE^{\frac{1}{p}}[\tilde{\xi}_1^{2p}]P^{\frac{1}{q}}\{|\tilde{\xi}_1| > \delta\} \le KnE^{\frac{1}{p}}[\tilde{\xi}_1^{2p}]h^{\frac{1}{q}}.$$

We now evaluate

$$E[\tilde{\xi}_1^{2p}] = E\Big[\Big(\frac{\xi_1}{\sqrt{nVar(\xi_1)}} - \frac{E[\xi_1]}{\sqrt{nVar(\xi_1)}}\Big)^{2p}\Big] \le KE\Big[\Big(\frac{\xi_1}{\sqrt{nVar(\xi_1)}}\Big)^{2p}\Big] + K\Big(\frac{E[\xi_1]}{\sqrt{nVar(\xi_1)}}\Big)^{2p}.$$

From the expression of $nE[\xi_1]$ above (35) and the given one for $\sqrt{nVar(\xi_1)}$ the last term equals

$$\Big(h^{\frac{1}{2}-u}\log^2\frac{1}{\varepsilon}\Big)^{2p}$$

On the other hand

$$E\Big[\frac{\xi_1^{2p}}{(nVar(\xi_1))^p}\Big] \le K\Big(\frac{E[(X_1X_2)^{2p}]}{(nVar(\xi_1))^p} + 2\frac{E[(hA_2)^{2p}X_1^{2p}]}{(nVar(\xi_1))^p} + \frac{E[(h^2A_1A_2)^{2p}]}{(nVar(\xi_1))^p}\Big):$$

the last term contributes with $\Big(h^{3/2-u}\log^2\frac{1}{\varepsilon}\Big)^{2p}$; the second term, by the Burkhölder-Davis-Gundy inequality and recalling that $\alpha_1 = 1$, is dominated by

$$K\left(\frac{h^2\log^2\frac{1}{\varepsilon}\int_{t_{i-1}}^{t_i}\int_0^{\varepsilon}x^2\nu^{(1)}(dx)}{h\varepsilon^2}\right)^p = \Big(h^{2-u}\log^2\frac{1}{\varepsilon}\Big)^p;$$

and the first term is

$$\frac{E[X_1^{2p}X_2^{2p}]}{(h\varepsilon^2)^p} = \frac{E^2[X_1^{2p}]}{(h\varepsilon^2)^p} \le K\frac{(h\varepsilon)^{2p}}{(h\varepsilon^2)^p} = Kh^p.$$

Thus

$$E[\tilde{\xi}_1^{2p}] \le K\Big(\Big(h^{3/2-u}\log^2\frac{1}{\varepsilon}\Big)^{2p} + \Big(h^{2-u}\log^2\frac{1}{\varepsilon}\Big)^p + h^p + \Big(h^{\frac{1}{2}-u}\log^2\frac{1}{\varepsilon}\Big)^{2p}\Big) \sim \Big(h^{\frac{1}{2}-u}\log^2\frac{1}{\varepsilon}\Big)^{2p}.$$

It follows that by choosing q sufficiently close to 1, and precisely $q < 1/(2u)$, we have

$$nE^{\frac{1}{p}}[\tilde{\xi}_1^{2p}]h^{\frac{1}{q}} \le Kn\Big(h^{\frac{1}{2}-u}\log^2\frac{1}{\varepsilon}\Big)^{2p\cdot\frac{1}{p}}h^{\frac{1}{q}} \sim Kh^{\frac{1}{q}-2u}\log^4\frac{1}{\varepsilon} \to 0.$$ □

5.3 Proof of the tools for Theorem 3.2

Under the assumptions of Theorem 3.2 the following Lemmas holds true.
Lemma 4.8 Let L be a one-sided α-stable process with characteristic triplet $(z, 0, c \cdot I_{\{x>0\}} x^{-1-\alpha} dx)$, let $H_1 \doteq (L_t - zt)_t$, take $\varepsilon = \varepsilon(h)$ s.t. $h/\varepsilon(h) \to 0$, any constant $p \in (0,1)$ s.t. $p > |z|h/\varepsilon$ and any $q \in (0, 1-p)$. For $m = 1, 2$, $i = 1..n$ we have the following.

1. $P\{\Delta_i N^{(m)} \neq 0, (\Delta_i M^{(m)})^2 > r_h\} \leq K\frac{h^2}{r_h}$, $P(|\Delta_i M^{(m)}| > K\sqrt{r_h}) \leq K\theta_m$.
2. $P\{|\Delta_i L| > \varepsilon, \sum_{s\in]t_{i-1},t_i]} I_{\{|\Delta L_s|>\varepsilon\}} = 0\} \leq K\tilde{\theta}^{4/3} + K\tilde{\theta}(q^{-\alpha} - 1)$.
3. $P\{|\Delta_i M^{(m)}| > \sqrt{r_h}(1-p), \Delta_i \tilde{V}^{(m)} = 0\} \leq K\theta_m^{4/3} + K\theta_m(q^{-\alpha_m} - 1)$.
4. $P\{|\Delta_i H_1| \leq \varepsilon(1+p), \sum_{s\in]t_{i-1},t_i]} I_{\{|\Delta H_{1s}|>\varepsilon\}} = 1\} \leq K[\tilde{\theta}^{4/3} + \tilde{\theta}(1-(1+2p)^{-\alpha})]$

   $P\{|\Delta_i L| \leq \varepsilon, \sum_{s\in]t_{i-1},t_i]} I_{\{|\Delta L_s|>\varepsilon\}} = 1\} \leq K[\tilde{\theta}^{4/3} + \tilde{\theta}(1-(1+2p)^{-\alpha})]$.
5. With $\varepsilon = \sqrt{r_h}$ we have $P\{|\Delta_i M^{(m)}| \leq \sqrt{r_h}(1+p), \Delta_i \tilde{V}^{(m)} \geq 1\} \leq K\theta_m^{4/3} + K\theta_m(1 - (1+2p)^{-\alpha_m})$.

Proof . Point 1. By the independence of $N^{(m)}$, $M^{(m)}$ and using the Markov inequality for $P\{(\Delta_i M^{(m)})^2 > r_h\}$, we reach $P\{\Delta_i N^{(m)} \neq 0, (\Delta_i M^{(m)})^2 > r_h\} \leq Kh\frac{h\int_0^1 x^2 \nu^{(m)}(dx)}{r_h} = K\frac{h^2}{r_h}$. The second inequality is a trivial consequence of Lemma 6 in [1], as $M^{(m)}$ is a semimartingale following the same model as $X^{(m)}$ in (1) with $a \equiv \sigma \equiv J^{(m)} \equiv 0$.

Point 2: the idea here is to look at H_1 as half of a symmetric stable process. More precisely, take an independent and identically distributed copy H_2 of H_1, then $\tilde{L} = H_1 - H_2$ is a symmetric α-stable process. Let us fix any p as in the assumptions and call $\tilde{L}'$, and H'_ℓ the processes $\tilde{L}$, H_ℓ deprived of their jumps bigger than ε, e.g. $H'_{\ell t} = H_{\ell t} - \sum_{s\leq t} \Delta H_{1s} I_{\{|\Delta H_{1s}|>\varepsilon\}}$. Note that if $|\Delta_i L| > \varepsilon$ then $|\Delta_i H_1| = |\Delta_i L - zh| > |\Delta_i L| - |z|h > \varepsilon - |z|h > \varepsilon(1-p)$, and also that the jumps of L and H_1 are the same and are positive, thus

$$P\Big\{|\Delta_i L| > \varepsilon, \sum_{s\in]t_{i-1},t_i]} I_{\{\Delta L_s>\varepsilon\}} = 0\Big\} \leq P\Big\{|\Delta_i H_1| > \varepsilon(1-p), \sum_{s\in]t_{i-1},t_i]} I_{\{\Delta H_{1s}>\varepsilon\}} = 0\Big\}$$

$$= P\Big\{|\Delta_i H'_1| > \varepsilon(1-p), \sum_{s\in]t_{i-1},t_i]} I_{\{\Delta H_{1s}>\varepsilon\}} = 0\Big\} \leq P\big\{|\Delta_i H'_1| > \varepsilon(1-p)\big\} \qquad (39)$$

$$= P\big\{|\Delta_i H'_1| > \varepsilon(1-p), \Delta_i H'_2 \leq \varepsilon(1-p-q)\big\} + P\big\{|\Delta_i H'_1| > \varepsilon(1-p), \Delta_i H'_2 > \varepsilon(1-p-q)\big\}.$$

Now on the first set of the last display we have $|\Delta_i \tilde{L}'| = |\Delta_i H'_1 - \Delta_i H'_2| > |\Delta_i H'_1| - |\Delta_i H'_2| > \varepsilon(1-p) - \varepsilon(1-p-q) = \varepsilon q$, while the probability of the second set, by the independence of the H'_ℓ, is $P\{|\Delta_i H'_1| > \varepsilon(1-p)\}P\{|\Delta_i H'_2| > \varepsilon(1-p-q)\}$ which is dominated by $K\tilde{\theta}^2$ by Lemma 6 in [1], applied with $a \equiv \sigma \equiv J \equiv 0$, $M_t = \int_0^t \int_0^\varepsilon x\tilde{\mu}(dx) - t\int_\varepsilon^1 x\nu(dx)$, ν the Lévy measure of L and $\tilde{\mu}$ the compensated jump measure of L. It follows that (39) is dominated by

$$P\{|\Delta_i \tilde{L}'| > \varepsilon q\} + K\tilde{\theta}^2 : \qquad (40)$$

note that $P\{|\Delta_i\tilde{L}'| > \varepsilon q\} = P\left\{|\Delta_i\tilde{L}'| > \varepsilon q, \Delta_i\tilde{\tilde{N}} = 0\right\} + P\left\{|\Delta_i\tilde{L}'| > \varepsilon q, \Delta_i\tilde{\tilde{N}} \geq 1\right\}$: by the independence of $\Delta_i\tilde{L}'$ on $\Delta_i\tilde{\tilde{N}}$ and using Remark 4.7, points 1 and 2, we have

$$P\left\{|\Delta_i\tilde{L}'| > \varepsilon q, \Delta_i\tilde{\tilde{N}} \geq 1\right\} = P\{|\Delta_i\tilde{L}'| > \varepsilon q\}P\left\{\Delta_i\tilde{\tilde{N}} \geq 1\right\} \leq K\tilde{\theta}^{7/3}.$$

Therefore (40) is dominated by $K\tilde{\theta}^{7/3} + P\{|\Delta_i\tilde{L}'| > \varepsilon q, \Delta_i\tilde{\tilde{N}} = 0\} + K\tilde{\theta}^2$; noting that $\tilde{\theta}^{7/3} << \tilde{\theta}^2$, since $q < 1$, the last display is dominated by

$$P\Big\{|\Delta_i\tilde{L}'| > \varepsilon q, \sum_{s\in]t_{i-1},t_i]} I_{\{|\Delta\tilde{L}_s|>\varepsilon q\}} = 0\Big\} + P\Big\{\sum_{s\in]t_{i-1},t_i]} I_{\{|\Delta\tilde{L}_s|\in(\varepsilon q,\varepsilon]\}} \geq 1\Big\} + K\tilde{\theta}^2 : \quad (41)$$

using Remark 4.7, point 2 with εq in place of ε, Remark 4.7 point 1 and the fact that $\tilde{\theta}^2 << \tilde{\theta}^{4/3}$, we reach our thesis.

Point 3 is a consequence of point 2. Let us denote $L^{(m)}$ with L. We have

$$P\{|\Delta_i M^{(m)}| > \sqrt{r_h}(1-p), \Delta_i\tilde{V}^{(m)} = 0\} = P\{|\Delta_i M^{(m)}| > \sqrt{r_h}(1-p), \Delta_i\tilde{V}^{(m)} = 0,$$

$$\sum_{s\in]t_{i-1},t_i]} I_{\{\Delta L_s>1\}} = 0\} + P\{|\Delta_i M^{(m)}| > \sqrt{r_h}(1-p), \Delta_i\tilde{V}^{(m)} = 0, \sum_{s\in]t_{i-1},t_i]} I_{\{\Delta L_s>1\}} \geq 1\}$$

$$\leq P\{|\Delta_i H_1'| > \sqrt{r_h}(1-p), \sum_{s\in]t_{i-1},t_i]} I_{\{\Delta H_{1s}>\sqrt{r_h}\}} = 0\} + P\{\sum_{s\in]t_{i-1},t_i]} I_{\{\Delta L_s>1\}} \geq 1\}$$

the first term is bounded by the one in (39) with $\varepsilon = \sqrt{r_h}$, while the second one involves the Poisson process counting the jumps of L bigger than 1 within $]t_{i-1}, t_i]$, which has parameter $hU(1)$, thus the thesis follows.

Point 4. With the same notations as at point 2, we have

$$P\{|\Delta_i H_1| \leq \varepsilon(1+p), \sum_{s\in]t_{i-1},t_i]} I_{\{|\Delta H_{1s}|>\varepsilon\}} = 1\} = P\{|\Delta_i H_1| \leq \varepsilon(1+p), \sum_{s\in]t_{i-1},t_i]} I_{\{|\Delta H_{1s}|>\varepsilon\}} = 1,$$

$$|\Delta_i H_2| > \varepsilon p\} + P\{|\Delta_i H_1| \leq \varepsilon(1+p), \sum_{s\in]t_{i-1},t_i]} I_{\{|\Delta H_{1s}|>\varepsilon\}} = 1, |\Delta_i H_2| \leq \varepsilon p\} : \quad (42)$$

the first term of the right hand side (rhs) is dominated by $P\{\sum_{s\in]t_{i-1},t_i]} I_{\{|\Delta H_{1s}|>\varepsilon\}} = 1, |\Delta_i H_2| > \varepsilon p\} = P\{\sum_{s\in]t_{i-1},t_i]} I_{\{|\Delta H_{1s}|>\varepsilon\}} = 1\}P\{|\Delta_i H_2| > \varepsilon p\} \leq K\tilde{\theta}^2$, having used the independence and Remark 4.7 points 1 and 2. As for the second term, on $\{|\Delta_i H_1| \leq \varepsilon(1+p), |\Delta_i H_2| \leq \varepsilon p\}$ we have $|\Delta_i\tilde{L}| = |\Delta_i H_1 - \Delta_i H_2| \leq |\Delta_i H_1| + |\Delta_i H_2| \leq \varepsilon(1+p) + \varepsilon p = \varepsilon(1+2p)$. Moreover, by their independence, the two H_ℓ have no common jumps, so a jump of H_2 cannot neutralize any jumps of H_1, thus $\sum_{s\in]t_{i-1},t_i]} I_{\{|\Delta H_{1s}|>\varepsilon\}} = 1 \Rightarrow \sum_{s\in]t_{i-1},t_i]} I_{\{|\Delta\tilde{L}_s|>\varepsilon\}} \geq 1$. Since $P\{\sum_{s\in]t_{i-1},t_i]} I_{\{|\Delta\tilde{L}_s|>\varepsilon\}} \geq 2\} \leq K\tilde{\theta}^2$, it follows that (42) is dominated by

$$K\tilde{\theta}^2 + P\{|\Delta_i\tilde{L}| \leq \varepsilon(1+2p), \sum_{s\in]t_{i-1},t_i]} I_{\{|\Delta\tilde{L}_s|>\varepsilon\}} = 1\}$$

$$\leq K\tilde{\theta}^2 + P\{|\Delta_i\tilde{L}| \leq \varepsilon(1+2p), \sum_{s\in]t_{i-1},t_i]} I_{\{|\Delta\tilde{L}_s|>\varepsilon(1+2p)\}} = 1\} + P\{\sum_{s\in]t_{i-1},t_i]} I_{\{|\Delta\tilde{L}_s|\in(\varepsilon,\varepsilon(1+2p))\}} = 1\}$$

and the first thesis follows by Remark 4.7 point 1, with $\varepsilon(1+2p)$ in place of ε, point 2 and the fact that $\tilde{\theta}^2 << \tilde{\theta}^{4/3}$. The second inequality at point 4 follows from the previous one. In fact if $|\Delta_i L| \le \varepsilon$ then $|\Delta_i H_1| = |\Delta_i L - zh| \le |\Delta_i L| + |z|h < \varepsilon(1+p)$, further L and H_1 do exactly the same jumps, thus

$$P\{|\Delta_i L| \le \varepsilon, \sum_{s\in]t_{i-1},t_i]} I_{\{|L_s|>\varepsilon\}} = 1\} \le P\{|\Delta_i H_1| \le \varepsilon(1+p), \sum_{s\in]t_{i-1},t_i]} I_{\{|H_{1s}|>\varepsilon\}} = 1\}.$$

Point 5 follows from point 4. Let us again denote $L^{(m)}$ with L. We have

$$P\{|\Delta_i M^{(m)}| \le \sqrt{r_h}(1+p), \Delta_i \tilde{V}^{(m)} \ge 1\} = \tag{43}$$

$$P\{|\Delta_i M^{(m)}| \le \sqrt{r_h}(1+p), \Delta_i \tilde{V}^{(m)} \ge 1, \sum_{s\in]t_{i-1},t_i]} I_{\{|\Delta L_s|>1\}} = 0\}+$$

$$P\{|\Delta_i M^{(m)}| \le \sqrt{r_h}(1+p), \Delta_i \tilde{V}^{(m)} \ge 1, \sum_{s\in]t_{i-1},t_i]} I_{\{|\Delta L_s|>1\}} \ge 1\} :$$

the second term of the rhs is bounded by Kh, as at Point 3. On the set at the first term the jumps of M coincide with the jumps of L, and the very M coincides with H_1. Thus the last display is dominated by

$$P\{|\Delta_i H_1| \le \sqrt{r_h}(1+p), \sum_{s\in]t_{i-1},t_i]} I_{\{|\Delta H_{1s}|>\sqrt{r_h}\}} \ge 1, \sum_{s\in]t_{i-1},t_i]} I_{\{|\Delta L_s|>1\}} = 0\} + Kh \le$$

$$P\{|\Delta_i H_1| \le \sqrt{r_h}(1+p), \sum_{s\in]t_{i-1},t_i]} I_{\{|\Delta H_{1s}|>\sqrt{r_h}\}} = 1\} + P\{\sum_{s\in]t_{i-1},t_i]} I_{\{|\Delta H_{1s}|>\sqrt{r_h}\}} \ge 2\} + Kh,$$

and the thesis follows by Lemma 4.8, point 4, Remark 4.7, point 1 and $Kh << \theta_m$. □

Lemma 4.9 Let, for i=1..n, $A_i \subset \Omega$ be independent on $W^{(1)}$ and $W^{(2)}$ and s.t. $\forall i, P(A_i) \le \theta_m$. If each $\sigma^{(j)}$ satisfy (6), then

i) $\frac{1}{\theta_m}\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \sigma_s^{(1)} dW_s^{(1)} \int_{t_{i-1}}^{t_i} \sigma_s^{(2)} dW_s^{(2)} I_{A_i} \sim \frac{1}{\theta_m}\sum_{i=1}^n \sigma_{t_{i-1}}^{(1)} \Delta_i W^{(1)} \sigma_{t_{i-1}}^{(2)} \Delta_i W^{(2)} I_{A_i}$.

ii) Any $P(A_i)$ is, we have $E[|\sum_{i=1}^n \sigma_{t_{i-1}}^{(1)} \Delta_i W^{(1)} \sigma_{t_{i-1}}^{(2)} \Delta_i W^{(2)} I_{A_i}|] \le KP(A_i)$.

Proof i) Denote $\sigma_i := \sigma_{t_i}$. We have $\sigma_s = \sigma_{i-1} + (\sigma_s - \sigma_{i-1})$, thus

$$\frac{1}{\theta_2}\sum_{i=1}^n \Big[\int_{t_{i-1}}^{t_i} \sigma_s^{(1)} dW_s^{(1)} \int_{t_{i-1}}^{t_i} \sigma_s^{(2)} dW_s^{(2)} - \sigma_{t_{i-1}}^{(1)} \Delta_i W^{(1)} \sigma_{t_{i-1}}^{(2)} \Delta_i W^{(2)}\Big] I_{A_i} = \tag{44}$$

$$\frac{1}{\theta_m}\sum_{i=1}^n \Big[\sigma_{i-1}^{(1)} \Delta_i W^{(1)} \int_{t_{i-1}}^{t_i} (\sigma_s^{(2)} - \sigma_{i-1}^{(2)}) dW_s^{(2)} + \int_{t_{i-1}}^{t_i} (\sigma_s^{(1)} - \sigma_{i-1}^{(1)}) dW_s^{(1)} \sigma_{i-1}^{(2)} \Delta_i W^{(2)} +$$

$$\int_{t_{i-1}}^{t_i} (\sigma_s^{(1)} - \sigma_{i-1}^{(1)}) dW_s^{(1)} \int_{t_{i-1}}^{t_i} (\sigma_s^{(2)} - \sigma_{i-1}^{(2)}) dW_s^{(2)}\Big] I_{A_i}.$$

Firstly note that

$$E\Big[\int_{t_{i-1}}^{t_i}(\sigma_s^{(m)}-\sigma_{i-1}^{(m)})^2 ds\,|\,A_i\Big]=\frac{E\Big[\int_{t_{i-1}}^{t_i}(\sigma_s^{(m)}-\sigma_{i-1}^{(m)})^2 ds I_{A_i}\Big]}{P(A_i)}\le\frac{E\Big[\int_{t_{i-1}}^{t_i}(\sigma_s^{(m)}-\sigma_{i-1}^{(m)})^2 ds\Big]}{P(A_i)}$$

is bounded by $Kh/P(A_i)$. It follows that

$$E\Big[|\sigma_{i-1}^{(1)}\Delta_i W^{(1)}\int_{t_{i-1}}^{t_i}(\sigma_s^{(2)}-\sigma_{i-1}^{(2)})dW_s^{(2)}|\Big]=E\Big[|\sigma_{i-1}^{(1)}\Delta_i W^{(1)}\int_{t_{i-1}}^{t_i}(\sigma_s^{(2)}-\sigma_{i-1}^{(2)})dW_s^{(2)}|\Big|A_i\Big]P(A_i)$$

$$\le\sqrt{E\Big[|\sigma_{i-1}^{(1)}\Delta_i W^{(1)}|^2\Big|A_i\Big]}\sqrt{E\Big[\Big(\int_{t_{i-1}}^{t_i}(\sigma_s^{(2)}-\sigma_{i-1}^{(2)})dW_s^{(2)}\Big)^2\Big|A_i\Big]}P(A_i); \qquad (45)$$

since $W^{(m)}$ is independent on A_i, its law under P and under $P(\cdot\mid A_i)$ is the same, thus it keeps its martingale property also under $P(\cdot\mid A_i)$. However, any bounded càdlàg integrand η is, the stochastic integral $\eta.W^{(m)}$ is a martingale under $P(\cdot|\,A_i)$, thus $E\Big[|\int_{t_{i-1}}^{t_i}\eta_s dW_s^{(m)}|^2\Big|A_i\Big]$ $=E\Big[\int_{t_{i-1}}^{t_i}\eta_s^2 ds\Big|A_i\Big]$, and (45) coincides with

$$\sqrt{E\Big[(\sigma_{i-1}^{(1)})^2 h\Big|A_i\Big]}\sqrt{E\Big[\int_{t_{i-1}}^{t_i}(\sigma_s^{(2)}-\sigma_{i-1}^{(2)})^2 ds\Big|A_i\Big]}P(A_i)\le K\sqrt{h}\sqrt{\int_{t_{i-1}}^{t_i}\frac{h}{P(A_i)}ds P(A_i)},$$

which equals $Kh\sqrt{hP(A_i)}$. Therefore the norm $||.||_1$ of the first term in the rhs of (44) is bounded by

$$\frac{1}{\theta_m}\sum_{i=1}^n E\Big[\Big|\sigma_{i-1}^{(1)}\Delta_i W^{(1)}\int_{t_{i-1}}^{t_i}(\sigma_s^{(2)}-\sigma_{i-1}^{(2)})dW_s^{(2)}\Big|I_{A_i}\Big]\le K\frac{1}{\theta_m}nh\sqrt{hP(A_i)}\le K\sqrt{\frac{h}{\theta_m}}\to 0.$$

We reach the same result also for the second term in the rhs of (44). Finally

$$\frac{1}{\theta_m}\sum_{i=1}^n E\Big[\Big|\int_{t_{i-1}}^{t_i}(\sigma_s^{(1)}-\sigma_{i-1}^{(1)})dW_s^{(1)}\int_{t_{i-1}}^{t_i}(\sigma_s^{(2)}-\sigma_{i-1}^{(2)})dW_s^{(2)}I_{A_i}\Big|\Big]\le$$

$$\frac{1}{\theta_m}\sum_{i=1}^n\sqrt{E\Big[\Big(\int_{t_{i-1}}^{t_i}(\sigma_s^{(1)}-\sigma_{i-1}^{(1)})dW_s^{(1)}\Big)^2\Big|A_i\Big]}\sqrt{E\Big[\int_{t_{i-1}}^{t_i}(\sigma_s^{(2)}-\sigma_{i-1}^{(2)})^2 ds\Big|A_i\Big]}P(A_i)\le$$

$$K\frac{1}{\theta_m}\sum_{i=1}^n\frac{h^2}{P(A_i)}P(A_i)\le K\varepsilon^{\alpha_m}\to 0.$$

ii) Similarly, $E[|\sum_{i=1}^n\sigma_{i-1}^{(1)}\Delta_i W^{(1)}\sigma_{i-1}^{(2)}\Delta_i W^{(2)}I_{A_i}|]=$
$\sum_{i=1}^n E\Big[|\sigma_{i-1}^{(1)}\Delta_i W^{(1)}\sigma_{i-1}^{(2)}\Delta_i W^{(2)}|\Big|A_i\Big]P(A_i)\le K\sum_{i=1}^n hP(A_i)\le KP(A_i).$ □

Lemma 4.10 With $\stackrel{ucp}{\to}$ denoting convergence in probability uniformly on $[0,T]$ and $IC_t \doteq \int_0^t \rho_s \sigma_s^{(1)} \sigma_s^{(2)} ds$, we have

$$\frac{1}{\theta_m} \sum_{i=1}^{[t/h]} \int_{t_{i-1}}^{t_i} \sigma_s^{(1)} dW_s^{(1)} \int_{t_{i-1}}^{t_i} \sigma_s^{(2)} dW_s^{(2)} I_{\{\Delta_i \tilde{V}^{(m)} \geq 1\}} \stackrel{ucp}{\to} \frac{c_m}{\alpha_m} IC_t.$$

Proof By the independence of each $W^{(j)}$ on $\tilde{V}^{(m)}$, using Lemma 4.9 and Remark 4.7 point 2, we have that the left hand side (lhs) of the above display has the same asymptotic behavior (in the $\sim$ sense) as

$$\frac{1}{\theta_m} \sum_{i=1}^{[t/h]} \sigma_{t_{i-1}}^{(1)} \Delta_i W^{(1)} \sigma_{t_{i-1}}^{(2)} \Delta_i W^{(2)} I_{\{\Delta_i \tilde{V}^{(m)} \geq 1\}} \doteq \sum_{i=1}^{[t/h]} \eta_i.$$

However we have

$$\sum_{i=1}^{[t/h]} E_{i-1}[\eta_i] = \frac{1}{\theta_m} \sum_{i=1}^{[t/h]} \sigma_{t_{i-1}}^{(1)} \sigma_{t_{i-1}}^{(2)} E_{i-1}[\Delta_i W^{(1)} \Delta_i W^{(2)}] P_{i-1}\{\Delta_i \tilde{V}^{(m)} \geq 1\} :$$

$E_{i-1}[\Delta_i W^{(1)} \Delta_i W^{(2)}] = E_{i-1}[\int_{t_{i-1}}^{t_i} \rho_s ds]$, and $P_{i-1}\{\Delta_i \tilde{V}^{(m)} \geq 1\} = 1 - e^{-\lambda_m h}$ with $\lambda_m = c_m \frac{r_h^{-\frac{\alpha_m}{2}}}{\alpha_m}$, and $|1 - e^{-\lambda_m h} - \lambda_m h| \leq K\theta_m^2$, thus the last display has the same limit in probability as

$$\frac{1}{\theta_m} \sum_{i=1}^{[t/h]} \sigma_{i-1}^{(1)} \sigma_{i-1}^{(2)} E_{i-1}\Big[\int_{t_{i-1}}^{t_i} \rho_s ds\Big] \lambda_m h. \tag{46}$$

Further, $\frac{1}{\theta_m} E\Big[\sum_{i=1}^{[t/h]} |\sigma_{t_{i-1}}^{(1)} \sigma_{t_{i-1}}^{(2)}| \cdot \Big| E_{i-1}\Big[\int_{t_{i-1}}^{t_i} \rho_s ds\Big] - \rho_{t_{i-1}}\Big| \cdot \lambda_m h\Big]$

$$\leq \frac{K}{\theta_m} E\Big[\sum_{i=1}^{[t/h]} |\sigma_{t_{i-1}}^{(1)} \sigma_{t_{i-1}}^{(2)}| E_{i-1}\Big[\int_{t_{i-1}}^{t_i} |\rho_s - \rho_{t_{i-1}}| ds\Big] \lambda_m h\Big] \leq \frac{Knh^2\theta_m}{\theta_m} \to 0,$$

and this implies that (46) has the same limit in probability as

$$\frac{1}{\theta_m} \sum_{i=1}^{[t/h]} \sigma_{t_{i-1}}^{(1)} \sigma_{t_{i-1}}^{(2)} \rho_{t_{i-1}} h \frac{c_m}{\alpha_m} \theta_m \stackrel{P}{\to} \frac{c_m}{\alpha_m} IC.$$

However by separating $\sigma_{i-1}^{(1)} \sigma_{i-1}^{(2)} \rho_{i-1} = (\sigma_{i-1}^{(1)} \sigma_{i-1}^{(2)} \rho_{i-1})^+ - (\sigma_{i-1}^{(1)} \sigma_{i-1}^{(2)} \rho_{i-1})^-$ and applying the reasoning indicated in [9], just before (3.5), we reach that such a convergence is also ucp. Further

$$\sum_{i=1}^{[t/h]} E_{i-1}[\eta_i^2] = \frac{1}{\theta_m^2} \sum_{i=1}^{[t/h]} (\sigma_{t_{i-1}}^{(1)} \sigma_{t_{i-1}}^{(2)})^2 E_{i-1}[(\Delta_i W^{(1)})^2 (\Delta_i W^{(2)})^2] P_{i-1}\{\Delta_i \tilde{V}^{(m)} \geq 1\} :$$

by using the expression of $W^{(2)}$ given after (1) we find that $E_{i-1}[(\Delta_i W^{(1)})^2(\Delta_i W^{(2)})^2] \le Kh^2$, thus

$$\sum_{i=1}^{[t/h]} E_{i-1}[\eta_i^2] \le K\frac{1}{\theta_m^2}h\theta_m \sum_{i=1}^{[t/h]}(\sigma^{(1)}_{t_{i-1}}\sigma^{(2)}_{t_{i-1}})^2 h \sim \frac{h}{\theta_m}\int_0^t (\sigma^{(1)}_s\sigma^{(2)}_s)^2 ds \sim \varepsilon^{\alpha_m} \to 0.$$

Thus, by Lemma 4.2 in [9], the thesis follows. □

Lemma 4.11 We have

$$\frac{1}{\theta_1}\sum_{i=1}^{[t/h]} \sigma^{(1)}_{t_{i-1}}\Delta_i W^{(1)}\sigma^{(2)}_{t_{i-1}}\Delta_i W^{(2)} I_{\{\Delta_i\tilde V^{(1)}\ge 1, \Delta_i\tilde V^{(2)}\ge 1\}} \stackrel{ucp}{\to} (1-\gamma)\frac{c_1}{\alpha_1}IC_t \cdot I_{\{\gamma\in[0,1)\}}.$$

Proof Let us start by proving that

$$P\{\Delta_i\tilde V^{(1)}\ge 1, \Delta_i\tilde V^{(2)}\ge 1\} \approx (1-\gamma)\theta_1\frac{c_1}{\alpha_1}I_{\{\gamma\in[0,1)\}}. \tag{47}$$

In fact, with $\varepsilon = \sqrt{r_h}$, such a probability equals $P\Big\{\mu(]t_{i-1},t_i]\times(\varepsilon,1]\times(\varepsilon,1])\ge 1\Big\} = 1-e^{-\lambda h} \approx \lambda h$, where $\lambda = \nu_\gamma(]t_{i-1},t_i]\times(\varepsilon,1]\times(\varepsilon,1])\ge 1\})$. In view of (3) and of the shape of $f(x)$ due to our choice of the parameters (see figure 1), we have $\lambda = (1-\gamma)\int_{(\varepsilon,1]\times(\varepsilon,1]} 1\nu_\|(dx_1,dx_2)$: $\nu_\|$ only weights the points (x_1,x_2) with $x_2 = f(x_1)$, and $x_1\wedge f(x_1) > \varepsilon$ means that $x_1 > f^{-1}(\varepsilon)\vee\varepsilon = \varepsilon$, while $x_1 \vee f(x_1)\le 1$ means that $x_1 \le f^{-1}(1)\wedge 1 = f^{-1}(1)$, thus $\lambda = (1-\gamma)\nu_1((\varepsilon, f^{-1}(1)]) = (1-\gamma)\Big[c_1\frac{\varepsilon^{-\alpha_1}}{\alpha_1} - \frac{c_2}{\alpha_2}\Big]$, having used that $f = U_2^{-1}\circ U_1$. However if $\gamma\ne 1$ then the leading term of $P\{\Delta_i\tilde V^{(1)}\ge 1, \Delta_i\tilde V^{(2)}\ge 1\}$ is $(1-\gamma)\theta_1\frac{c_1}{\alpha_1}$. If $\gamma = 1$ then $\lambda = 0$, and (47) is verified.

Let us now define

$$\sum_{i=1}^{[t/h]}\frac{1}{\theta_1}\sigma^{(1)}_{t_{i-1}}\Delta_i W^{(1)}\sigma^{(2)}_{t_{i-1}}\Delta_i W^{(2)} I_{\{\Delta_i\tilde V^{(1)}\ge 1, \Delta_i\tilde V^{(2)}\ge 1\}} = \sum_{i=1}^{[t/h]}\chi_i :$$

by the independence of each $W^{(m)}$ on each $\tilde V^{(\ell)}$, we have

$$\sum_{i=1}^{[t/h]} E_{i-1}[\chi_i] \approx \sum_{i=1}^{[t/h]}\sigma^{(1)}_{t_{i-1}}\sigma^{(2)}_{t_{i-1}}E_{i-1}\Big[\int_{t_{i-1}}^{t_i}\rho_s ds\Big](1-\gamma)\frac{c_1}{\alpha_1}I_{\{\gamma\in[0,1)\}} \stackrel{P}{\to} (1-\gamma)\frac{c_1}{\alpha_1}IC_t\cdot I_{\{\gamma\in[0,1)\}};$$

as in the previous Lemma, we reach that such a convergence is also ucp. Further, $\sum_{i=1}^{[t/h]} E_{i-1}[\chi_i^2] \le K\frac{h\theta_1}{\theta_1^2} \le K\varepsilon^{\alpha_1}\to 0$, so, by Lemma 4.2 in [9], the thesis follows. □

References

[1] Y. Aït-Sahalia and J. Jacod. Estimating the degree of activity of jumps in high frequency data. *Annals of Statistics*, 37:2202–2244, 2009.

[2] Y. Aït-Sahalia and J. Jacod. Testing whether jumps have finite or infinite activity. *Annals of Statistics*, 39:1689–1719, 2011.

[3] T.G. Andersen, T. Bollerslev, and F.X. Diebold. Roughing it up: Including jump components in the measurement, modeling and forecasting of return volatility. *The Review of Economics and Statistics, MIT Press*, 89(4):701–720, 2007.

[4] K. Boudt, C. Croux, and S. Laurent. Outlyingness weighted covariation. *Journal of Financial Econometrics*, 9:657–684, 2010.

[5] R. Cont and C. Mancini. Nonparametric tests for pathwise properties of semimartingales. *Bernoulli*, 17(2):781–813, 2011.

[6] R. Cont and P. Tankov. *Financial Modelling with Jump Processes*. CRC Press, 2004.

[7] F. Gobbi and C. Mancini. Diffusion covariation and co-jumps in bidimensional asset price processes with stochastic volatility and infinite activity Lévy jumps. *www.ArXiv.org*, 2013.

[8] J. Jacod. Asymptotic properties of realized power variations and associated functions of semimartingales. *Stochastic Processes and Their Applications*, 118:517–559, 2008.

[9] J. Jacod. *Statistical Methods for Stochastic Differential Equations*, chapter Statistics and high-frequency data. Chapman and Hall/CRC, 2012.

[10] J. Jacod and M Reiss. A remark on the rates of convergence for integrated volatility estimation in the presence of jumps. *Annals of Statistics*, 42(3):1131–1144, 2014.

[11] J. Jacod and V. Todorov. Efficient estimation of integrated volatility in presence of infinite variation jumps. *Annals of Statistics*, 42:1029–1069, 2014.

[12] J. Kallsen and P. Tankov. Characterization of dependence of multidimensional lévy processes using lévy copulas. *Journal of Multivariate Analysis*, 97:1551–1572, 2006.

[13] C. Mancini. Non-parametric threshold estimation for models with stochastic diffusion coefficient and jumps. *Scandinavian Journal of Statistics*, 36:270–296, 2009.

[14] C. Mancini. The speed of convergence of the threshold estimator of integrated variance. *Stochastic processes and their applications*, 121(4):845–855, 2011.

[15] C. Mancini. Measuring the relevance of the microstructure noise in observed financial data. *Stochastic Processes and their Applications*, 123:2728–2751, 2013.

[16] C. Mancini and F. Gobbi. Identifying the brownian covariation from the co-jumps given discrete observations. *Econometric Theory*, 28(2):249–273, 2006.

[17] K. Sato. *Lévy processes and infinitely divisible distributions*. Cambridge University Press, 1999.

[18] P. Tankov. Dependence structure of lévy processes with applications in risk management. *Internal report 502 of CMAP, Ecole Polytechnique*, 2003.